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Symplectic Fermions

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Abstract

We study two-dimensional conformal field theories generated from a “symplectic fermion” — a free two-component fermion field of spin one — and construct the maximal local supersymmetric conformal field theory generated from it. This theory has central charge $c = -2$ and provides the simplest example of a theory with logarithmic operators.

Twisted states with respect to the global $SL(2, \mathbb{C})$ -symmetry of the symplectic fermions are introduced and we describe in detail how one obtains a consistent set of twisted amplitudes. We study orbifold models with respect to finite subgroups of $SL(2, \mathbb{C})$ and obtain their modular invariant partition functions. In the case of the cyclic orbifolds explicit expressions are given for all two-, three- and four-point functions of the fundamental fields. The \mathcal{C}_2 -orbifold is shown to be isomorphic to the bosonic local logarithmic conformal field theory of the triplet algebra encountered previously. We discuss the relation of the \mathcal{C}_4 -orbifold to critical dense polymers.

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1 Introduction

Over the last few years it has become apparent that there is a class of conformal field theories whose correlation functions have logarithmic branch cuts. Such conformal field theories are believed to be important for the description of certain statistical models, in particular in the theory of (multi)critical polymers and percolation [1–5], two-dimensional turbulence [6–8], and the quantum Hall effect [9, 10]. Models analysed so far include

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the WZNW model on the supergroup $GL(1,1)$ [11], the coset model $SL(2, \mathbb{C})/SU(2)$ [10], the $c = -2$ model [12, 13], gravitationally dressed conformal field theories [14] and some critical disordered models [15–19]. Singular vectors of some Virasoro models have been constructed in [20], correlation functions have been calculated in [21, 22], and more structural properties of logarithmic conformal field theories have been analysed in [23].

The class of chiral logarithmic conformal field theories is also interesting from a conceptual point of view. There exist logarithmic models which behave in many respects like ordinary (non-logarithmic) chiral conformal field theories, and it is not yet clear in which way these models differ structurally from conventional theories. They include the models investigated in this paper, a series of “quasirational logarithmic” Virasoro models [13] and a series of “rational logarithmic” models, the simplest of which is the triplet algebra at $c = -2$ [24]. Here quasirational means that a countable set of representations of the chiral algebra closes under fusion (with finite fusion rules), and rational that the same holds for a certain finite set of representations, including all (finitely many) irreducible representations. For rational models Zhu’s algebra [25] is finite dimensional, and it should be possible to read off all properties of the whole chiral theory from the vacuum representation. In particular, one should be able to decide *a priori* whether the chiral algebra leads to a logarithmic theory or not, without actually constructing all the amplitudes. As yet little progress has been made in this direction, although it is believed that unitary (rational) meromorphic conformal field theories always lead to non-logarithmic theories.

The only rational logarithmic model which has been studied in detail so far, the aforementioned triplet algebra at $c = -2$, possesses another oddity (apart from the appearance of indecomposable reducible representations which lead to logarithmic correlation functions), and it is quite possible that this is true in more generality [26]: although the theory possesses a finite fusion algebra, the matrices corresponding to the reducible representations cannot be diagonalised, and a straightforward application of Verlinde’s formula does not make sense. This is mirrored by the fact that the modular transformation properties of some of the characters cannot be described by constant matrices as they depend on the modular parameter τ . This might suggest that these logarithmic rational theories only make sense as chiral theories, and that they do not correspond to modular invariant (non-chiral) conformal field theories. It was demonstrated in [27] that, at least for the case of the triplet algebra at $c = -2$, this is not the case. The resulting theory is in every aspect a standard (non-chiral) conformal field theory but for the property that it does not factorise into standard chiral theories. Among other things, this demonstrates that a non-chiral conformal field theory has significantly more structure than the two chiral theories it is built from.

The general strategy for constructing a (non-chiral) conformal field theory is as follows. Determine the two-, three- and four-point functions of the *fundamental* fields (the fields that correspond to the fundamental states of the different representations). Given the two- and three-point functions of the fundamental fields, all other amplitudes can be derived from these, and the consistency conditions of all amplitudes can be reduced to those being obeyed by the four-point functions. This reduces the problem of constructing any amplitude to a finite computation which can be done in principle. All data of a conformal

field theory can then be recovered from the complete set of amplitudes, see [28]. Due to the complex structure of the non-chiral representations for the triplet model it was not feasible in [27] to complete this programme. However, the two- and three-point functions agree with those of a model built from a two-component free fermion field, the “symplectic fermion” model. Since the two- and three-point amplitudes uniquely determine the theory, consistency of the symplectic fermion model implies consistency of the triplet model.

Our strategy in this paper mirrors the one in [27]: Firstly we construct the representations of the chiral algebra generated by the symplectic fermions. In this case we obtain a unique maximal indecomposable non-chiral representation. The non-chiral amplitudes are then determined as co-invariants with respect to the comultiplication for the fermion field. After imposing the consistency constraints on the hierarchy of amplitudes we arrive at a unique local logarithmic conformal field theory \mathcal{SF} generated from the symplectic fermions. This theory admits a global $SL(2, \mathbb{C})$ symmetry under which the fermions transform in the fundamental representation. We can thus introduce twisted states such that the symplectic fermions acquire a phase when moved along a closed loop around a twisted state. Amplitudes involving twisted states are still co-invariants with respect to the (twisted) comultiplication for the symplectic fermions and we shall use this to explicitly calculate the local two-, three- and four-point amplitudes for twist fields. The consistency constraints give a system of quadratic equations in the three-point couplings which admits a simple solution.

Orbifold models $\mathcal{SF}[\mathcal{G}]$ with respect to a finite subgroup of $SL(2, \mathbb{C})$ are obtained by restricting to (twisted and untwisted) states which are invariant under \mathcal{G} . For the cyclic subgroups this can be done explicitly with the local orbifold amplitudes given by the above hierarchy of twisted amplitudes where all fields are \mathcal{G} -invariant. We determine the different sectors of the models and give their modular invariant partition functions. The \mathcal{C}_2 orbifold model is the logarithmic conformal field theory for the triplet algebra discussed in [27] while the \mathcal{C}_4 orbifold model is related to dense critical polymers [1]. Our method cannot be applied to the non-abelian orbifolds, for these we can only determine the partition functions by path-integral arguments.

This paper is organised as follows. In Section 2 we define the symplectic fermion model and construct its non-chiral amplitudes. In Section 3 we introduce twisted sectors and find their amplitudes. Finally, in Section 4 we investigate the various orbifold models with respect to finite subgroups of the global $SL(2, \mathbb{C})$ symmetry: the cyclic groups in sections 4.1 and 4.2 and non-abelian orbifolds in section 4.3. The equivalence of the \mathcal{C}_2 orbifold with the triplet model is established in Section 4.4 and the relation of the \mathcal{C}_4 -orbifold to critical dense polymers is discussed in Section 4.5. The more technical aspects of the paper have been relegated to a number of appendices. The locality of the symplectic fermion model is established in Appendix A. The twisted amplitudes are explicitly constructed in Appendix B. The character of the orbifold chiral algebra is derived in Appendix C.

2 Symplectic fermions

We take as our starting point the (η, ξ) ghost system with the first order action [29]

$$S = \frac{1}{\pi} \int d^2z (\eta \bar{\partial} \xi + \bar{\eta} \partial \bar{\xi}) \quad (1)$$

where η and ξ are conjugate *fermion* fields of dimensions 1 and 0, respectively. The operator product of the chiral fields has short-distance limit

$$\eta(z)\xi(w) \sim \frac{1}{z-w}, \quad \xi(z)\eta(w) \sim \frac{1}{z-w} \quad (2)$$

up to terms regular as $z \rightarrow w$. The stress tensor for this system has central charge $c = -2$ and reads

$$T(z) = :\partial \xi(z) \eta(z):, \quad (3)$$

where $:\cdots:$ indicates fermionic normal ordering. The system also possesses a natural $U(1)$ current given by

$$J(z) = :\xi(z) \eta(z):, \quad (4)$$

which counts ξ with charge +1 and η with charge -1.

Viewed as a conformal field theory by itself this system has a number of problematic features. If one wishes the vacuum state Ω to be translation invariant the current is not a primary field [29], rather one has

$$T(z)J(w) \sim \frac{-1}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}. \quad (5)$$

As a consequence one has charge asymmetry, $J_0^\dagger = 1 - J_0$. This implies anomalous charge conservation in vacuum expectation values,

$$\langle \Omega | \psi_1(z_1, \bar{z}_1) \cdots \psi_n(z_n, \bar{z}_n) | \Omega \rangle = 0, \quad \text{unless } \Sigma q_i = \Sigma \bar{q}_i = 1, \quad (6)$$

where q_i and \bar{q}_i are the charges of ψ_i with respect to the currents J and \bar{J} . In terms of the fermions this means that vacuum expectation values are non-vanishing only if they contain exactly one unpaired ξ and $\bar{\xi}$ field. One has two ground states, the invariant vacuum Ω and $\xi_0 \bar{\xi}_0 \Omega$, with vanishing norms but $\langle \Omega | \xi_0 \bar{\xi}_0 | \Omega \rangle = 1$. Furthermore it is not possible to construct an inner product on the space states compatible with the standard hermiticity properties of the stress tensor, $L_n^\dagger = L_{-n}$. A further problem is the appearance of fields which are neither primary fields nor descendants of a primary field, the simplest examples of which are ξ and J ; for further details see [3].

The ghost system can be bosonised by defining $J(z) = \partial \phi(z)$ in which case the stress tensor takes on the Feigin-Fuchs form. However, the aforementioned problems still persist. In the Coulomb gas construction of the Virasoro minimal models they are resolved by defining the physical states through a BRST resolution [30]. This identifies the two ground states and removes the reducible representations of the Virasoro algebra leaving

only the field content of the minimal model. The central charge $c = -2$ falls into the pattern $c = 1 - 6(p - q)^2/(pq)$ for the minimal series albeit with parameters $p = 1, q = 2$ outside the allowed range. The screening charge for the BRST resolution corresponds to η_0 and the usual “physical” space $\ker \eta_0 / \text{im } \eta_0$ is trivial, as observed in [1]. This is reflected in the fact that the standard Dotsenko-Fateev correlators [31] vanish identically when evaluated for $c = -2$ due to cancellations between the conformal blocks [1]. One can get finite results by considering the limit $c \rightarrow -2$ however the algebraic structure of the Virasoro representations is discontinuous [13].

It was noted in [29] that the (η, ξ) system contains an irreducible chiral algebra \mathcal{A} generated by η and $\partial\xi$. Both of these are Virasoro primary fields of dimension one. We can put them on an equal footing by grouping them together as a two-component fermion field χ^α of dimension one. We shall now apply the algebraic methods introduced in [27] to construct a local conformal field theory, the “symplectic fermion model” \mathcal{SF} , containing the chiral algebra \mathcal{A} generated by χ^α . The resulting model is generated by a *non-chiral* two-component free fermion field θ^α of dimension zero such that $\chi^\alpha = \partial\theta^\alpha, \bar{\chi}^\alpha = \bar{\partial}\theta^\alpha$. In contrast, the (η, ξ) system can be obtained by adjoining to $\mathcal{A} \otimes \bar{\mathcal{A}}$ the two *chiral* fields ξ and $\bar{\xi}$. The two models thus correspond to different local slices of the same non-local theory.

2.1 Chiral structure

The chiral algebra \mathcal{A} of the symplectic fermion model is generated by a two-component fermion field,

$$\chi^\alpha(z) = \sum_{n \in \mathbb{Z}} \chi_n^\alpha z^{-n-1}, \quad (7)$$

of conformal weight one. The field has short-distance expansion

$$\chi^\alpha(z) \chi^\beta(w) \sim \frac{d^{\alpha\beta}}{(z-w)^2} \quad (8)$$

resulting in anti-commutators for the Fourier modes,

$$\{\chi_m^\alpha, \chi_n^\beta\} = m d^{\alpha\beta} \delta_{m+n}, \quad (9)$$

where $d^{\alpha\beta}$ is an anti-symmetric tensor; we may choose a basis χ^\pm such that $d^{+-} = 1$. This algebra has a *unique* irreducible highest weight representation. Its highest weight state Ω is annihilated by all non-negative fermion modes, $\chi_m^\alpha \Omega = 0$ for $m \geq 0$. This representation is isomorphic to \mathcal{A} and provides its vacuum representation with Ω as the vacuum state. The chiral algebra contains a Virasoro algebra \mathcal{Vir} of central charge $c = -2$ given by

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \frac{1}{2} d_{\alpha\beta} : \chi^\alpha(z) \chi^\beta(z) :, \quad (10)$$

where $d_{\alpha\beta}$ is the inverse of $d^{\alpha\beta}$ such that $d^{\alpha\beta} d_{\beta\gamma} = \delta_\gamma^\alpha$. In algebraic language the stress tensor corresponds to the conformal state

$$L_{-2} \Omega = \frac{1}{2} d_{\alpha\beta} \chi_{-1}^\alpha \chi_{-1}^\beta \Omega. \quad (11)$$

The vacuum Ω is Möbius invariant under this Virasoro algebra, $L_m\Omega = 0$ for $m > -2$.

While the irreducible representation is unique it can be extended to obtain reducible but indecomposable representations. Denote by \mathcal{A}^\sharp the maximal generalised highest weight representation of \mathcal{A} obtained as such an the extension of the vacuum representation. It is freely generated by the negative modes, χ_m^α for $m < 0$, from a four dimensional space of ground states. This space is spanned by two bosonic states Ω and ω , and two fermionic states, θ^α , with the action of the zero-modes χ_0^α given by

$$\begin{aligned}\chi_0^\alpha \omega &= -\theta^\alpha, \\ \chi_0^\alpha \theta^\beta &= d^{\alpha\beta} \Omega, \\ L_0 \omega &= \Omega.\end{aligned}\tag{12}$$

The states ω and Ω span a two-dimensional Jordan block for L_0 . A conformal field theory based on this maximal representation will thus contain logarithmic operators.

2.2 Non-chiral theory

To construct the non-chiral theory we proceed as described in [27]. Details of the locality constraints are presented in Appendix A. The result is that there is a *unique* non-chiral symplectic fermion model \mathcal{SF} . It has two bosonic ground states, Ω, ω , and two fermionic ground states θ^α , satisfying

$$\begin{aligned}\chi_0^\alpha \omega &= \bar{\chi}_0^\alpha \omega = -\theta^\alpha, \\ \chi_0^\alpha \theta^\beta &= \bar{\chi}_0^\alpha \theta^\beta = d^{\alpha\beta} \Omega, \\ L_0 \omega &= \bar{L}_0 \omega = \Omega.\end{aligned}\tag{13}$$

The full non-chiral space of states \mathcal{W} is generated by the free action of the negative modes, $\chi_m^\alpha, \bar{\chi}_m^\alpha$ with $m < 0$, from the ground states. Using the comultiplication (68), amplitudes of excited states can be expressed in terms of amplitudes involving only fundamental fields, *i.e.* fields corresponding to the four ground states. The amplitudes of up to four fundamental fields are

$$\begin{aligned}\langle \omega \rangle &= 1, \\ \langle \omega \omega \rangle &= -2(\circ - \circ), \\ \langle \omega \omega \omega \rangle &= 2(\circ - \circ - \circ) - (\circ = \circ), \\ \langle \omega \omega \omega \omega \rangle &= 2(\circ - \circ - \circ - \circ) - 2(\circ = \circ - \circ) - 2(\nabla), \\ \langle \theta^\alpha \theta^\beta \rangle &= d^{\alpha\beta}, \\ \langle \theta^\alpha \theta^\beta \omega \rangle &= -d^{\alpha\beta} (\Delta_{13} + \Delta_{23} - \Delta_{12}), \\ \langle \theta^\alpha \theta^\beta \omega \omega \rangle &= d^{\alpha\beta} \left[(\Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{24} - 2\Delta_{12} - \Delta_{34}) \Delta_{34} \right. \\ &\quad \left. - (\Delta_{13} - \Delta_{14})(\Delta_{23} - \Delta_{24}) \right], \\ \langle \theta^\alpha \theta^\beta \theta^\gamma \theta^\delta \rangle &= d^{\alpha\beta} d^{\gamma\delta} (\Delta_{12} + \Delta_{34}) - d^{\alpha\gamma} d^{\beta\delta} (\Delta_{13} + \Delta_{24}) + d^{\alpha\delta} d^{\beta\gamma} (\Delta_{14} + \Delta_{23}),\end{aligned}\tag{14}$$

where

$$\begin{aligned}
(\circ-\circ) &= \sum_{ij} \Delta_{ij}, & (\circ=\circ) &= \sum_{ij} \Delta_{ij}^2, \\
(\circ-\circ-\circ) &= \sum_{ijk} \Delta_{ij} \Delta_{jk}, & (\circ=\circ-\circ) &= \sum_{ijkl} \Delta_{ij}^2 \Delta_{kl}, \\
(\circ-\circ-\circ-\circ) &= \sum_{ijkl} \Delta_{ij} \Delta_{jk} \Delta_{kl}, & (\nabla) &= \sum_{ijk} \Delta_{ij} \Delta_{jk} \Delta_{ki}
\end{aligned}$$

and

$$\Delta_{ij} \equiv \Delta(z_{ij}) = \mathcal{Z} + \ln |z_{ij}|^2. \quad (15)$$

The parameter \mathcal{Z} could be set to zero by redefining $\omega \mapsto \omega + \mathcal{Z}\Omega$, but it will be convenient later on to retain this parameter.

The sums are over pairwise distinct labelled graphs (*i.e.* graphs with vertices); labelled graphs that differ by a graph symmetry are only counted once. In amplitudes, Ω acts as the unit operator, except that its one-point function vanishes, $\langle \Omega \rangle = 0$. The relevant operator product expansions are, to lowest order in x ,

$$\begin{aligned}
\theta^\alpha(x) \theta^\beta &\sim d^{\alpha\beta} (\omega + \Delta(x)\Omega), \\
\theta^\alpha(x) \omega &\sim -\Delta(x) \theta^\alpha, \\
\omega(x) \omega &\sim -\Delta(x) (2\omega + \Delta(x)\Omega).
\end{aligned} \quad (16)$$

The operator product of Ω with any field S is simply given by $\Omega(x)S = S$ to all orders.

The structure of the theory is simple enough to allow us to determine not just the four-point amplitudes but all $2N$ -point amplitudes of the θ fields.

$$\langle \theta^{\alpha_1} \dots \theta^{\alpha_{2N}} \rangle = \frac{1}{2^{2N}} \sum_{\sigma \in S_{2N}} \text{sign}(\sigma) d^{\alpha_1 \alpha_2} d^{\alpha_3 \alpha_4} \dots d^{\alpha_{2N-1} \alpha_{2N}} \Delta_{\sigma_3 \sigma_4} \dots \Delta_{\sigma_{2N-1} \sigma_{2N}} \quad (17)$$

where the sum is over all permutations of the fields and the prefactor ensures that each grouping of the $2N$ fields into N pairs is only counted once. This can be proved by induction: the amplitudes satisfy the correct differential equations in the z_i and are thus fixed up to addition of a constant; considering the OPE $\theta^\alpha \theta^\beta$ fixes that constant. By contracting two θ fields one can obtain amplitudes involving ω . In particular, the N -point amplitude for ω can be written a sum over graphs of $N-1$ links between the N vertices,

$$\langle \omega \dots \omega \rangle = \sum_{g \in \mathcal{G}_N^{N-1}} (-1)^{N+\mathcal{N}_c} 2^{\mathcal{N}_d} \Delta(g), \quad (18)$$

where the propagator $\Delta(g)$ for the graph g is given as the product of a propagator Δ_{ij} for each link between vertices i and j , \mathcal{N}_c is the number of components including the isolated vertex \circ and the two-vertex loop $\circ=\circ$ while \mathcal{N}_d is the number of components excluding \circ and $\circ=\circ$. The combinatorial factor arises from contracting the $d^{\alpha\beta}$ tensors in the $2N$ -point amplitude $\langle \theta \dots \theta \rangle$ with the $d_{\alpha\beta}$ tensors in the OPE of $\theta^\alpha \theta^\beta$ and factors as the product of the following factors for each component of the graph g ,

$$\begin{array}{ll}
\circ & 1 \\
\circ=\circ & -1 \\
\circ-\circ-\dots-\circ-\circ & 2(-1)^{n+1} \\
\underbrace{\circ-\circ-\dots-\circ-\circ} & 2(-1)^{n+1}
\end{array}$$

Here the last two subgraphs contain n vertices. For example, the five- and six-point amplitudes of ω can be written as

$$\begin{aligned}
\langle \omega \omega \omega \omega \omega \rangle &= 2(\circ - \circ - \circ - \circ - \circ) - 2(\underbrace{\circ - \circ - \circ - \circ}_{\text{subgraph}}) + (\circ = \circ \circ = \circ) \\
&\quad - 4(\circ - \circ \underbrace{\circ - \circ - \circ}_{\text{subgraph}}) - 2(\circ - \circ - \circ \circ = \circ) \\
\langle \omega \omega \omega \omega \omega \omega \rangle &= -2(\circ - \circ - \circ - \circ - \circ - \circ) + 2(\underbrace{\circ - \circ - \circ - \circ - \circ}_{\text{subgraph}}) - 2(\circ = \circ \underbrace{\circ - \circ - \circ}_{\text{subgraph}}) \\
&\quad + 4(\circ - \circ \underbrace{\circ - \circ - \circ - \circ}_{\text{subgraph}}) + 4(\circ - \circ - \circ \underbrace{\circ - \circ - \circ}_{\text{subgraph}}) - 2(\circ - \circ \circ - \circ \circ = \circ) \\
&\quad + 2(\circ - \circ - \circ - \circ \circ = \circ)
\end{aligned}$$

These amplitudes define a fully consistent local conformal field theory on the Riemann sphere. This does not imply that the theory can be defined consistently on higher genus Riemann surfaces. Indeed, the partition function of this model is not modular invariant. To define the theory on the torus one has to add additional sectors where the fermion fields satisfy anti-periodic boundary conditions along the fundamental cycles of the torus — the different spin structures. This will be discussed in more detail in Section 4.

3 Twisted sectors

The symplectic fermion model admits a global $SL(2, \mathbb{C})$ symmetry under which the fermion fields χ^α transform in the fundamental representation. As in any theory where we have a global group of automorphisms \mathcal{G} we can introduce twist fields τ_g for any $g \in \mathcal{G}$, such that, as a field ϕ from \mathcal{W} is taken around the insertion of a twist field τ_g we obtain the original field up to the action of the automorphism g ,

$$\phi(e^{2\pi i} z, e^{-2\pi i} \bar{z}) \tau_g = (g\phi)(z, \bar{z}) \tau_g. \quad (19)$$

For the (chiral and anti-chiral) symplectic fermions this means,

$$\chi^\alpha(e^{2\pi i} z) \tau_g = (g\chi^\alpha)(z) \tau_g, \quad \bar{\chi}^\alpha(e^{2\pi i} \bar{z}) \tau_g = (g^{-1}\bar{\chi}^\alpha)(\bar{z}) \tau_g. \quad (20)$$

We shall restrict to the case where $\mathcal{G} \subset U(1)$ is an *abelian* subgroup of $SL(2, \mathbb{C})$. In this case all the twists commute and we can choose a basis χ^\pm for the two fermion fields such that $d^{+-} = 1$ and the automorphisms are of the form

$$g: \chi^\pm \mapsto e^{\pm 2\pi i \lambda} \chi^\pm \quad (21)$$

for some λ and we can use that parameter λ to label the different twisted sectors. It also follows that the symplectic fermions have a mode expansion

$$\chi^\pm(z) = \sum_{n \in \mathbb{Z}} \chi_{-n \mp \lambda}^\pm z^{n-1 \pm \lambda}, \quad \bar{\chi}^\pm(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{\chi}_{-n \pm \lambda}^\pm \bar{z}^{n-1 \mp \lambda} \quad (22)$$

when acting on the λ -twisted sector. Denote by \mathcal{W}_λ the representation freely generated from a ground state μ_λ satisfying $\chi_r^\alpha \mu_\lambda = \bar{\chi}_r^\alpha \mu_\lambda = 0$ for $r > 0$. This twisted representation

is irreducible since χ^\pm does not have zero-modes. The ground state μ_λ is a Virasoro highest weight state with conformal weight

$$h_\lambda = -\frac{\lambda(1-\lambda)}{2}. \quad (23)$$

In this expression, and below, we always take $0 < \lambda < 1$. We also write $\lambda^* = 1 - \lambda$. We will now construct the amplitudes for the twisted sectors. We need to determine only the amplitudes of the cyclic states ω and μ_λ since they determine all other amplitudes through the twisted comultiplication (79). As usual, the system of differential equations arising from the (twisted) comultiplication determines the amplitudes up to some structure constants; the details can be found in Appendix B. Since the overall twist of an amplitude has to vanish, all amplitudes with a single twist field vanish and the non-vanishing amplitudes with two twist fields are

$$\begin{aligned} \langle \mu_\lambda \mu_{\lambda^*} \rangle &= -\mathcal{O}_\lambda |z_{12}|^{2\lambda\lambda^*}, \\ \langle \mu_\lambda \mu_{\lambda^*} \omega \rangle &= \mathcal{O}_\lambda |z_{12}|^{2\lambda\lambda^*} \left(\mathcal{Z}_\lambda + \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right), \\ \langle \mu_\lambda \mu_{\lambda^*} \omega \omega \rangle &= -\mathcal{O}_\lambda |z_{12}|^{2\lambda\lambda^*} \left[\left(\mathcal{Z}_\lambda + \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right) \left(\mathcal{Z}_\lambda + \ln \left| \frac{z_{14} z_{24}}{z_{12}} \right|^2 \right) - H^\lambda(x) H^{\lambda^*}(x) \right], \end{aligned} \quad (24)$$

where $H^\lambda(x)$ is given by

$$H^\lambda(x, \bar{x}) = -2\Re \left(\frac{(1-x)^\lambda}{\lambda} {}_2F_1(1, \lambda; 1+\lambda; 1-x) \right) + \mathcal{Y}_\lambda. \quad (25)$$

For each pair of twisted sectors we have two parameters, \mathcal{O}_λ and \mathcal{Z}_λ . The constant \mathcal{Y}_λ is fixed by locality of $\langle \mu_\lambda \mu_{\lambda^*} \omega \omega \rangle$ and given in Appendix B.1. Amplitudes involving three twist fields are

$$\begin{aligned} \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \rangle &= \mathcal{C}_{\lambda_1, \lambda_2, \lambda_3} \left| z_{12}^{\lambda_1 \lambda_2} z_{13}^{\lambda_1 \lambda_3} z_{23}^{\lambda_2 \lambda_3} \right|^2 \quad \text{for } \lambda_1 + \lambda_2 + \lambda_3 = 1, \\ &= \mathcal{C}_{\lambda_1, \lambda_2, \lambda_3} \left| z_{12}^{\lambda_1^* \lambda_2^*} z_{13}^{\lambda_1^* \lambda_3^*} z_{23}^{\lambda_2^* \lambda_3^*} \right|^2 \quad \text{for } \lambda_1 + \lambda_2 + \lambda_3 = 2, \end{aligned} \quad (26)$$

from which we obtain the OPEs

$$\begin{aligned} \mu_{\lambda_1}(x) \mu_{\lambda_2} &= -\frac{\mathcal{C}_{\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2}}{\mathcal{O}_{\lambda_1+\lambda_2}} |x|^{2\lambda_1 \lambda_2} \mu_{\lambda_1+\lambda_2} + \dots \quad \text{for } 0 < \lambda_1 + \lambda_2 < 1, \\ &= -\frac{\mathcal{C}_{\lambda_1, \lambda_2, 2-\lambda_1-\lambda_2}}{\mathcal{O}_{\lambda_1+\lambda_2-1}} |x|^{2\lambda_1^* \lambda_2^*} \mu_{\lambda_1+\lambda_2-1} + \dots \quad \text{for } 1 < \lambda_1 + \lambda_2 < 2, \\ &= -\frac{\mathcal{O}_{\lambda_1}}{\mathcal{O}} |x|^{2\lambda_1 \lambda_2} (\omega + \Delta^{(\lambda)}(x) \Omega) + \dots \quad \text{for } \lambda_1 + \lambda_2 = 1, \end{aligned} \quad (27)$$

where

$$\Delta^{(\lambda)}(x) = \ln |x|^2 + 2\mathcal{Z} - \mathcal{Z}_\lambda. \quad (28)$$

Amplitudes involving four twist fields can be expressed in terms of hypergeometric functions. As the expressions are quite complicated they are listed in Appendix B.2. By contracting two fields in these four-point amplitudes we can express the four-point structure constants in terms of the three-point couplings.

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1:$$

$$\mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} = \frac{\mathcal{C}_{\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2} \mathcal{C}_{\lambda_1+\lambda_2, \lambda_3, \lambda_4}}{\mathcal{O}_{\lambda_1+\lambda_2}}$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 3:$$

$$\mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} = \frac{\mathcal{C}_{\lambda_1, \lambda_2, 2-\lambda_1-\lambda_2} \mathcal{C}_{\lambda_1+\lambda_2-1, \lambda_3, \lambda_4}}{\mathcal{O}_{\lambda_1+\lambda_2-1}}$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2:$$

$$\begin{aligned} \mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} &= \frac{\mathcal{C}_{\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2} \mathcal{C}_{\lambda_1+\lambda_2, \lambda_3, \lambda_4}}{\mathcal{O}_{\lambda_1+\lambda_2} \sqrt{\rho(\lambda_1, \lambda_2) \rho(\lambda_3^*, \lambda_4^*)}} && \text{if } \lambda_1 + \lambda_2 < 1 \\ &= \frac{\mathcal{C}_{\lambda_3, \lambda_4, 1-\lambda_3-\lambda_4} \mathcal{C}_{\lambda_3+\lambda_4, \lambda_1, \lambda_2}}{\mathcal{O}_{\lambda_3+\lambda_4} \sqrt{\rho(\lambda_3, \lambda_4) \rho(\lambda_1^*, \lambda_2^*)}} && \text{if } \lambda_1 + \lambda_2 > 1 \\ &= \frac{\mathcal{O}_{\lambda_1} \mathcal{O}_{\lambda_3}}{\mathcal{O}} && \text{if } \lambda_1 + \lambda_2 = 1 \end{aligned}$$

Here, $\rho(\lambda, \lambda')$ is a ratio of gamma functions,

$$\rho(\lambda, \lambda') = \frac{\Gamma(1-\lambda-\lambda') \Gamma(\lambda) \Gamma(\lambda')}{\Gamma(\lambda+\lambda') \Gamma(1-\lambda) \Gamma(1-\lambda')}, \quad (29)$$

and $\mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}$ is the overall normalisation of the amplitude $\langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle$. Locality also fixes the parameter

$$\mathcal{Z}_\lambda = \mathcal{Z} + \vartheta_{\lambda, \lambda^*}, \quad \vartheta_{\lambda, \lambda'} = 2\psi(1) - \psi(\lambda) - \psi(\lambda'), \quad (30)$$

where $\psi(x) = \Gamma(x)'/\Gamma(x)$ is the digamma function. The structure constants $\mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}$ are completely symmetric in their arguments and considering all possible orderings yields quadratic constraints on the three-point couplings. A solution to these locality constraints is given by

$$\mathcal{O}_\lambda = 1, \quad \mathcal{C}_{\lambda_1, \lambda_2, \lambda_3} = \mathcal{C}_{\lambda_1^*, \lambda_2^*, \lambda_3^*} = \sqrt{\frac{\Gamma(\lambda_1) \Gamma(\lambda_2) \Gamma(\lambda_3)}{\Gamma(\lambda_1^*) \Gamma(\lambda_2^*) \Gamma(\lambda_3^*)}}, \quad (31)$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 1$. The four-point couplings are

$$\mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} = \mathcal{F}_{\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*} = \sqrt{\frac{\Gamma(\lambda_1) \Gamma(\lambda_2) \Gamma(\lambda_3) \Gamma(\lambda_4)}{\Gamma(\lambda_1^*) \Gamma(\lambda_2^*) \Gamma(\lambda_3^*) \Gamma(\lambda_4^*)}}, \quad (32)$$

for $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ and $\mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} = 1$ for $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2$.

This hierarchy of twisted amplitudes provides a consistent set of semi-local amplitudes: By construction, the ground states of the twisted sectors are \mathcal{G} -invariant and amplitudes involving only those the ground states and the cyclic field ω are local and satisfy all consistency constraints. However, excited states may not be \mathcal{G} -invariant and their amplitudes will acquire phase factors when moving fields along closed loops around such an excited field. To obtain local amplitudes we have to restrict to \mathcal{G} -invariant states; this defines the orbifold model $\mathcal{SF}[\mathcal{G}]$.

4 Orbifold models

An abelian subgroup \mathcal{G} of $SL(2, \mathbb{C})$ is either $U(1)$ or \mathcal{C}_N , the cyclic group of order N . In all these cases irreducible representations of \mathcal{G} are one-dimensional and can be labelled by a weight (or twist) λ such that the generator g acts as $\exp(2\pi i\lambda)$, as has been done implicitly in the previous section. In the case of \mathcal{C}_N the twists are $\lambda \in \mathbb{Z}/N$ while for $U(1)$ the twists are continuous, $\lambda \in \mathbb{R}$. In both cases the set of allowed twists can be identified with \mathcal{G} since shifting a twist by an integer results in the same representation.

Given the symplectic fermion model \mathcal{SF} with space of states \mathcal{W} and chiral algebra \mathcal{A} we construct the orbifold model $\mathcal{SF}[\mathcal{G}]$ as follows [32]. The space of states is given by the \mathcal{G} -invariant subspaces $\mathcal{H}_\lambda = \mathcal{W}_\lambda[\mathcal{G}]$ of the (twisted) modules \mathcal{W}_λ for all $\lambda \in \mathcal{G}$. They form representations of the orbifold chiral algebra $\mathcal{A}[\mathcal{G}]$. The orbifold amplitudes, given by the above twisted amplitudes where all fields are \mathcal{G} -invariant, are fully local. The spaces \mathcal{H}_λ are in general reducible representations of the orbifold chiral algebra $\mathcal{A}[\mathcal{G}]$ and decompose as

$$\mathcal{H}_\lambda = \bigoplus_{\mu \in \mathcal{G}} \mathcal{H}_{\lambda, \mu}. \quad (33)$$

We first consider the $\mathcal{Vir} \times U(1)$ characters of the spaces \mathcal{W} and \mathcal{W}_λ . The characters of the \mathcal{G} -invariant subspaces \mathcal{H}_0 and $\mathcal{H}_{\lambda, \mu}$ can then be obtained by specialising the $U(1)$ characters to the finite group \mathcal{G} and using the orthogonality of \mathcal{G} -characters.

Since \mathcal{W}_λ is irreducible with respect to the original chiral algebra \mathcal{A} it is simply the product of a chiral and an anti-chiral twisted sector,

$$\mathcal{W}_\lambda = \mathcal{V}_\lambda \otimes \bar{\mathcal{V}}_{\lambda^*}, \quad (34)$$

where \mathcal{V}_λ is generated from a ground state of conformal weight $h_\lambda = -\lambda\lambda^*/2$ by the modes $\chi_{-n-\lambda}^+$ and $\chi_{-n+\lambda-1}^-$ with $n \geq 0$. These chiral spaces decompose under \mathcal{G} as

$$\mathcal{V}_\lambda = \bigoplus_{\mu \in \mathcal{G}} \mathcal{V}_\lambda^\mu, \quad (35)$$

such that g acts as $\exp(2\pi i\mu)$ on \mathcal{V}_λ^μ . The spaces \mathcal{V}_λ^μ are modules of the orbifold chiral algebra $\mathcal{A}[\mathcal{G}]$ and we have chosen the ground state to be invariant under \mathcal{G} . We can introduce the chiral $\mathcal{Vir} \times U(1)$ character

$$\begin{aligned} \chi_{\mathcal{V}_\lambda}(\tau, g) &= \text{tr}_{\mathcal{V}_\lambda} (e^{2\pi i\tau(L_0 - c/24)} g) \\ &= q^{-\frac{\lambda\lambda^*}{2} + \frac{1}{12}} \prod_{n=0}^{\infty} (1 + gq^{n+\lambda})(1 + g^{-1}q^{n+\lambda^*}) \\ &= \eta(\tau)^{-1} \sum_{m \in \mathbb{Z}} g^m q^{(2m+2\lambda-1)^2/8} \end{aligned} \quad (36)$$

where $\eta(\tau)$ is the Dedekind η -function, $q = \exp(2\pi i\tau)$ and g is an element of $U(1)$. In the last line we used the Jacobi triple product identity. The untwisted sector \mathcal{W} is not

simply the product of a chiral and an anti-chiral representation. However, since it is freely generated from the ground states by the action of the chiral and anti-chiral algebras, \mathcal{A} and $\bar{\mathcal{A}}$, we have the non-chiral $\mathcal{V}ir \times U(1)$ character

$$\begin{aligned}
\chi_{\mathcal{W}}(\tau, g) &= \text{tr}_{\mathcal{V}_\lambda} \left(e^{2\pi i \tau (L_0 - c/24)} e^{-2\pi i \bar{\tau} (\bar{L}_0 - c/24)} g \right) \\
&= (q\bar{q})^{\frac{1}{12}} (2 + g + g^{-1}) \left(\prod_{n=1}^{\infty} (1 + gq^n)(1 + g^{-1}q^n) \right) \left(\prod_{n=1}^{\infty} (1 + g\bar{q}^n)(1 + g^{-1}\bar{q}^n) \right) \\
&= \left(q^{\frac{1}{12}} \prod_{n=0}^{\infty} (1 + gq^n)(1 + g^{-1}q^{n+1}) \right) \left(\bar{q}^{\frac{1}{12}} \prod_{n=0}^{\infty} (1 + g\bar{q}^{n+1})(1 + g^{-1}\bar{q}^n) \right) \\
&= |\eta(\tau)|^{-2} \left(\sum_{m \in \mathbb{Z}} g^m q^{(2m-1)^2/8} \right) \left(\sum_{m \in \mathbb{Z}} g^m \bar{q}^{(2m+1)^2/8} \right) \tag{37}
\end{aligned}$$

4.1 \mathcal{C}_{2N} orbifold

Specialising now to the case of $\mathcal{G} = \mathcal{C}_{2N}$ we have $g^{2N} = 1$ for any group element g and the twists are of the form $\lambda = l/2N$. Thus the character of $\mathcal{V}_{l/2N}$ can be written in the form

$$\chi_{\mathcal{V}_{\frac{l}{2N}}}(\tau, g) = \sum_{k=0}^{2N-1} g^k \Lambda_{2Nk-N+l, 2N^2}(\tau), \tag{38}$$

where $\Lambda_{n,m}(\tau) = \Theta_{n,m}(\tau)/\eta(\tau)$ with the classical theta function defined as

$$\Theta_{n,m}(\tau) = \sum_{k \in \mathbb{Z} + n/2m} q^{mk^2}. \tag{39}$$

The orbifold representations $\mathcal{H}_\lambda = \mathcal{W}_\lambda[\mathcal{C}_{2N}]$ with $\lambda \neq 0$ are reducible with respect to the orbifold chiral algebra $\mathcal{A}[\mathcal{C}_{2N}]$,

$$\mathcal{H}_\lambda = \bigoplus_{\mu \in \mathcal{C}_{2N}} \mathcal{H}_{\lambda, \mu} = \bigoplus_{\mu \in \mathcal{C}_{2N}} \mathcal{V}_\lambda^\mu \otimes \bar{\mathcal{V}}_{\lambda^*}^{\mu*}. \tag{40}$$

The non-chiral character of \mathcal{H}_λ^μ is just the modulus squared of the chiral character of \mathcal{V}_λ^μ ,

$$\chi_{\mathcal{H}_{l/2N, k/2N}}(\tau) = \left| \chi_{\mathcal{V}_{l/2N}^{k/2N}}(\tau) \right|^2 = \left| \Lambda_{2Nk-N+l, 2N^2}(\tau) \right|^2. \tag{41}$$

For the untwisted sector we obtain

$$\begin{aligned}
\chi_{\mathcal{H}_0}(\tau) &= |\eta(\tau)|^{-2} \sum_{m-\bar{m} \equiv 0(2N)} q^{(2m-1)^2/8} \bar{q}^{(2\bar{m}-1)^2/8} \\
&= |\eta(\tau)|^{-2} \sum_{l=0}^{2N-1} \left(\sum_{k \in \mathbb{Z}} q^{(4Nk+2l-1)^2/8} \right) \left(\sum_{\bar{k} \in \mathbb{Z}} \bar{q}^{(4N\bar{k}+2l-1)^2/8} \right) \\
&= \sum_{l=0}^{2N-1} \left| \Lambda_{(2l-1)N, 2N^2}(\tau) \right|^2 \tag{42}
\end{aligned}$$

The space of states of the orbifold theory $\mathcal{SF}[\mathcal{C}_{2N}]$ consists of the indecomposable (extended) vacuum module \mathcal{H}_0 , arising from the untwisted sector \mathcal{W} , and $2N(2N-1)$ sectors

$$\mathcal{H}_{\lambda,\mu} = \mathcal{V}_\lambda^\mu \otimes \bar{\mathcal{V}}_{\lambda^*}^{\mu*}, \quad (43)$$

with $\lambda \neq 0$, arising from the decomposition

$$\mathcal{H}_\lambda = \mathcal{W}_\lambda[\mathcal{C}_{2N}] = \bigoplus_{\mu \in \mathcal{C}_{2N}} \mathcal{H}_{\lambda,\mu}. \quad (44)$$

of the twisted sectors \mathcal{W}_λ . The partition function of $\mathcal{SF}[\mathcal{C}_{2N}]$ is thus

$$\begin{aligned} Z[\mathcal{C}_{2N}] &= \chi_{\mathcal{H}_0}(\tau) + \sum_{l=1}^{2N-1} \sum_{k=0}^{2N-1} \chi_{\mathcal{H}_{l/2N,k/2N}}(\tau) \\ &= \sum_{l=0}^{2N-1} \sum_{k=0}^{2N-1} |\Lambda_{2Nk-N+l,2N^2}(\tau)|^2 \\ &= \sum_{m=0}^{4N^2-1} |\Lambda_{m,2N^2}(\tau)|^2. \end{aligned} \quad (45)$$

This partition function is invariant under the modular group and is in fact identical to the partition function of a free boson compactified on a circle of radius $r = 2N$ using the normalisation of [33]. Furthermore, defining as in [34] the Coulomb gas partition function at radius ρ as

$$Z(\rho, \tau) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{4}(m\rho+n/\rho)^2} \bar{q}^{\frac{1}{4}(m\rho-n/\rho)^2} \quad (46)$$

we have $Z[\mathcal{C}_{2N}](\tau) = Z(\sqrt{2}N, \tau)$.

The orbifold chiral algebra $\mathcal{A}[\mathcal{C}_{2N}]$ has character (see Appendix C)

$$\chi_{\mathcal{A}[\mathcal{C}_{2N}]}(\tau) = \frac{1}{2N} \left[\eta(\tau)^2 + \sum_{l=0}^{N-1} (-1)^l (2N-2l-1) \Lambda_{(2l+1)N,2N^2}(\tau) \right] \quad (47)$$

Because of $\eta(-1/\tau)^2 = -\tau\eta(\tau)^2$, this does not transform nicely under the modular group. This is an indication that the chiral algebra $\mathcal{A}[\mathcal{C}_{2N}]$ should not by itself form a sector of the theory but rather be contained within the larger vacuum module \mathcal{H}_0 .

We can also decompose the chiral algebra \mathcal{A} with respect to the Virasoro algebra of central charge $c = -2$ contained within it. The character of \mathcal{A} can then be expressed in terms of the irreducible characters of the Virasoro algebra at $c = -2$ as

$$\chi_{\mathcal{A}}(\tau, u) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n u^{2l-n} \right) \chi_{n+1,1}^{\mathcal{V}_{ir}}(\tau), \quad (48)$$

and thus the character of the orbifold chiral algebra is given in terms of Virasoro characters as

$$\chi_{\mathcal{A}[\mathcal{C}_{2N}]}(\tau, u) = \sum_{k=0}^{\infty} \left(1 + 2 \left\lfloor \frac{k}{N} \right\rfloor \right) \chi_{2k+1,1}^{\mathcal{V}_{ir}}(\tau), \quad (49)$$

where $\lfloor x \rfloor$ is the largest integer less or equal to x . Viewed as a W -algebra, the orbifold chiral algebra $\mathcal{A}[\mathcal{C}_{2N}]$ is generated by the Virasoro field L of conformal weight two, a Virasoro primary field of weight three and a pair of Virasoro primary fields of weight $N(2N+1)$. In the case of $N=1$ the three Virasoro primary fields all have weight three and we obtain the triplet algebra, see Section 4.4.

4.2 \mathcal{C}_N orbifold

We can repeat the same procedure for \mathcal{C}_N orbifolds with N odd. However, the \mathcal{C}_N -invariant states still contain fermions. The partition function $Z[\mathcal{C}_N]$ corresponds to a single spin-structure and is not modular invariant. The \mathcal{C}_N orbifold has an additional \mathcal{C}_2 symmetry arising from the boson-fermion selection rule. Performing the sum over spin-structures in the $\mathcal{SF}[\mathcal{C}_N]$ model is the same as directly performing the orbifold construction with respect to $\mathcal{C}_N \times \mathcal{C}_2 = \mathcal{C}_{2N}$.

4.3 Non-abelian Orbifolds

The remaining finite subgroups of $SL(2, \mathbb{C})$ are the binary tetrahedral, octahedral and icosahedral groups \mathcal{T} , \mathcal{O} and \mathcal{I} and one would like to construct $\mathcal{SF}[\mathcal{G}]$ orbifolds for them as well. The general structure of orbifold models was investigated in [32]: The twisted sectors are labelled by conjugacy classes of \mathcal{G} . They can be constructed as before and, in fact, the structure of the g -twisted module depends only on the order of g . The n -point amplitudes are labelled by n -tuples of group elements (g_1, \dots, g_n) such that $g_1 \cdots g_n = 1$. Labels related by simultaneous conjugation yield the same amplitude. Our method for constructing the amplitudes relies on being able to simultaneously diagonalise the twist for all fields in the amplitude. This is not possible if some of the g_i are not mutually commuting as is necessarily the case for some amplitudes when \mathcal{G} is non-abelian. A direct construction of the \mathcal{T} , \mathcal{O} and \mathcal{I} orbifolds is thus not possible.

However, we can determine the torus partition function of $\mathcal{SF}[\mathcal{G}]$ as a sum over partition functions of the symplectic fermion field θ^α with different boundary conditions: For $g, h \in \mathcal{G}$ we denote the partition function of the h -twisted sector with an insertion of the operator g as

$$g \square_h = \text{tr}_{\mathcal{W}_h} \left(g q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right). \quad (50)$$

The \mathcal{G} -orbifold partition function is then obtained as

$$Z[\mathcal{G}] = \frac{1}{|\mathcal{G}|} \sum_{\substack{g, h \in \mathcal{G} \\ gh=hg}} g \square_h. \quad (51)$$

For non-abelian \mathcal{G} boundary conditions twisted by non-commuting group elements are not consistent, hence the condition $gh = hg$. This allows us to interpret the twisted partition functions as the $\mathcal{V}ir \times U(1)$ characters introduced previously,

$$g \square_h = \chi_{\mathcal{W}_h}(\tau, g). \quad (52)$$

For \mathcal{C}_{2N} we recover the previous result which we denote by $Z_N = Z[\mathcal{C}_{2N}]$. To calculate the partition function $Z[\mathcal{G}]$ for non-abelian \mathcal{G} we follow [35] and add the contributions of the mutually commuting subsets of \mathcal{G} , which form cyclic groups, subtracting any overcounting. As in [35] the result is

$$Z[\mathcal{D}_N] = \frac{1}{2}(Z_N + 2Z_2 - Z_1), \quad (53)$$

$$Z[\mathcal{T}] = \frac{1}{2}(2Z_3 + Z_2 - Z_1), \quad (54)$$

$$Z[\mathcal{O}] = \frac{1}{2}(Z_4 + Z_3 + Z_2 - Z_1), \quad (55)$$

$$Z[\mathcal{I}] = \frac{1}{2}(Z_5 + Z_3 + Z_2 - Z_1). \quad (56)$$

Here, of course, one has to keep in mind that $Z_N(\tau)$ corresponds to a Coulomb gas partition function at radius $N\sqrt{2}$ and not N as for $c = 1$.

4.4 The triplet model

We will now show that the \mathcal{C}_2 -twisted model is the logarithmic conformal field theory of [27]. The chiral algebra of the theory in [27] is a W -algebra, the so-called triplet algebra [36, 3]; it is realised in the symplectic fermion model by

$$\begin{aligned} L_{-2}\Omega &= \frac{1}{2}d_{\alpha\beta}\chi_{-1}^\alpha\chi_{-1}^\beta\Omega, \\ W_{-3}^a\Omega &= t_{\alpha\beta}^a\chi_{-2}^\alpha\chi_{-1}^\beta\Omega. \end{aligned} \quad (57)$$

Here the matrices $(t^a)_\beta^\alpha$ form the spin 1/2 representation of $sl(2)$ in a Cartan-Weyl basis,

$$t_\pm^{0\pm} = \pm\frac{1}{2}, \quad t_\pm^{\pm\mp} = 1, \quad (58)$$

with all other entries vanishing. The bosonic sector $\mathcal{H}_0 = \mathcal{W}_0[\mathcal{C}_2]$ of \mathcal{W}_0 contains the representation \mathcal{R} of the triplet algebra [27]. Explicitly, the ground states Ω and ω are identified in both representations and the higher level states of \mathcal{R} can be expressed as fermionic descendents as

$$\begin{aligned} \rho^{\alpha\bar{\alpha}} &= \chi_{-1}^\alpha\bar{\theta}^{\bar{\alpha}}, \\ \bar{\rho}^{\alpha\bar{\alpha}} &= -\bar{\chi}_{-1}^{\bar{\alpha}}\theta^\alpha, \\ \psi^{\alpha\bar{\alpha}} &= \chi_{-1}^\alpha\bar{\chi}_{-1}^{\bar{\alpha}}\Omega, \\ \phi^{\alpha\bar{\alpha}} &= \chi_{-1}^\alpha\bar{\chi}_{-1}^{\bar{\alpha}}\omega. \end{aligned} \quad (59)$$

On the other hand, the bosonic sector has the non-chiral character

$$\chi_{\mathcal{H}_0}(\tau) = 2 |\Lambda_{1,2}(\tau)|^2 = \chi_{\mathcal{R}}(\tau). \quad (60)$$

Since \mathcal{H}_0 and \mathcal{R} have the same character but \mathcal{H}_0 contains \mathcal{R} they are in fact identical, $\mathcal{H}_0 = \mathcal{R}$. By the same character argument we find that the orbifold algebra $\mathcal{A}[\mathcal{C}_2]$ is identical to the triplet algebra.

The other two representations of the triplet model, the irreducible representations $\mathcal{V}_{-1/8,-1/8}$ and $\mathcal{V}_{3/8,3/8}$, also have an interpretation in terms of the symplectic fermion theory: they correspond to the bosonic sector $\mathcal{H}_{\frac{1}{2}} = \mathcal{W}_{\frac{1}{2}}[\mathcal{C}_2]$ of the (unique) \mathcal{C}_2 -twisted representation $\mathcal{W}_{\frac{1}{2}}$. In this sector, the fermions are half-integrally moded, but all bosonic operators (including the triplet algebra generators that are bilinear in the fermions) are still integrally moded, and the twisted sector decomposes as

$$\mathcal{H}_{\frac{1}{2}} = \mathcal{H}_{\frac{1}{2},0} \oplus \mathcal{H}_{\frac{1}{2},\frac{1}{2}}. \quad (61)$$

with respect to the triplet algebra. The ground state μ of the twisted sector is identified with the ground state of $\mathcal{V}_{-1/8,-1/8}$ while the ground state of $\mathcal{V}_{3/8,3/8}$ is given as a fermionic descendant in $\mathcal{H}_{\frac{1}{2}}$,

$$\nu^{\alpha\bar{\alpha}} = \chi_{-\frac{1}{2}}^{\alpha} \bar{\chi}_{\frac{1}{2}}^{\bar{\alpha}} \mu. \quad (62)$$

Thus, $\mathcal{V}_{-1/8,-1/8}$ is contained in $\mathcal{H}_{\frac{1}{2},0}$ and $\mathcal{V}_{3/8,3/8}$ is contained in $\mathcal{H}_{\frac{1}{2},\frac{1}{2}}$ and by considering the characters,

$$\begin{aligned} \chi_{\mathcal{H}_{\frac{1}{2},0}}(\tau) &= |\Lambda_{0,2}(\tau)|^2 = \chi_{\mathcal{V}_{-1/8,-1/8}}(\tau), \\ \chi_{\mathcal{H}_{\frac{1}{2},\frac{1}{2}}}(\tau) &= |\Lambda_{2,2}(\tau)|^2 = \chi_{\mathcal{V}_{3/8,3/8}}(\tau), \end{aligned} \quad (63)$$

we find that, in fact, $\mathcal{H}_{\frac{1}{2},0} = \mathcal{V}_{-1/8,-1/8}$ and $\mathcal{H}_{\frac{1}{2},\frac{1}{2}} = \mathcal{V}_{3/8,3/8}$ as representations of the triplet algebra. The \mathcal{C}_2 orbifold and the triplet theory have the same partition function

$$Z[\mathcal{C}_2] = \sum_{k=0}^3 |\Lambda_{k,2}(\tau)|^2 = Z_{\text{triplet}}. \quad (64)$$

The amplitudes determined in Section 3 agree with those in [27] on setting $\mathcal{Z} = 4 \ln 2$ and thus $\mathcal{Z}_{1/2} = 8 \ln 2$. Furthermore, all three-point functions of the fundamental fields, $\rho^{\alpha\bar{\alpha}}, \bar{\rho}^{\alpha\bar{\alpha}}, \psi^{\alpha\bar{\alpha}}$ and $\phi^{\alpha\bar{\alpha}}$, of the triplet model agree with the corresponding excited amplitudes in the \mathcal{C}_2 -twisted symplectic fermion model. Both models are therefore isomorphic. This is the argument used in [27] to establish the consistency of the logarithmic theory presented there.

4.5 Critical dense polymers

Many properties of polymers in solution can be modeled by considering simple geometrical systems. In particular, dense polymers are obtained by considering a finite number of self-avoiding and mutually-avoiding loops or chains on a lattice that cover a finite fraction of

the available volume. It was argued in [1] that their continuum limit should correspond to a (ξ, η) system. Specifically, the (ξ, η) system on a torus with periodic or anti-periodic boundary conditions describes the sector formed by an even number of non-contractible loops while the sector formed by an odd number of non-contractible loops corresponds to \mathbb{Z}_4 -twisted boundary conditions. The total polymer partition function is equal to the partition function $Z[\mathcal{C}_4]$. This result was obtained by realising dense polymers as the $n \rightarrow 0$ limit of the low temperature phase of the $O(n)$ model which in turn can be mapped onto a Coulomb gas. This also reproduces the scaling dimensions

$$x_L^D = \frac{L^2 - 4}{16} \quad (65)$$

for the geometric polymer L -leg operators Φ_L . However, the limit $n \rightarrow 0$ does not commute with the thermodynamic limit or with changing the boundary conditions. Furthermore, the physical quantities which have been determined for dense polymers, the partition function $Z[\mathcal{C}_4]$ and the scaling dimensions (65), are shared by the \mathcal{C}_4 orbifold models of both the (ξ, η) system and the symplectic fermions. Both models necessarily involve reducible but indecomposable representations of the Virasoro algebra. The structure of these representations is quite different in the two models resulting in differences for the correlators. In particular, some amplitudes such as the four-point amplitude of the Ramond ground states of dimensions $-1/8$ vanish when calculated directly in the (ξ, η) system or in its Coulomb gas formulation. To get a non-zero result one has to take the generic Coulomb gas amplitude and take the limit $c \rightarrow -2$ resulting in an amplitude with logarithmic short-distance behaviour. This agrees with the amplitude $\langle \mu_{1/2} \mu_{1/2} \mu_{1/2} \mu_{1/2} \rangle$ for the symplectic fermions. The same is also the case for the other logarithmic amplitudes thus providing evidence that the correct description of the continuum theory of dense polymers is in terms of the $\mathcal{SF}[\mathcal{C}_4]$ model. However, a more detailed analysis is clearly needed: The 1-leg operator Φ_1 is two-fold degenerate and is represented by $\mu_{1/4}$ and $\mu_{3/4}$. The reason for this degeneracy given in [1] is that sources and sinks of polymers are distinguished. The four-point amplitude of the 1-leg operator Φ_1 (eq. (88) of [1]) agrees with the amplitude $\langle \mu_{1/4} \mu_{3/4} \mu_{1/4} \mu_{3/4} \rangle$ of the symplectic fermion model. However, we also obtain a non-vanishing amplitude for four sources (or four sinks),

$$\langle \mu_{1/4} \mu_{1/4} \mu_{1/4} \mu_{1/4} \rangle = \langle \mu_{3/4} \mu_{3/4} \mu_{3/4} \mu_{3/4} \rangle = \frac{\Gamma(1/4)^2}{\Gamma(3/4)^2} |z_{12} z_{13} z_{14} z_{23} z_{24} z_{34}|^{1/8}.$$

An important problem is to understand the role played by the logarithmic fields and identify their equivalents in the lattice realisation. In both the \mathbb{Z}_4 -twisted (ξ, η) -system and the orbifold model $\mathcal{SF}[\mathcal{C}_4]$ there are two states of conformal dimension $h = \bar{h} = 0$. According to [1] these correspond to the identity and the density operator ρ . In the (ξ, η) -system the density operator is represented by $\rho = \xi + \bar{\xi}$ and results in non-vanishing expectation value $\langle \rho \rangle$ and zero density-density correlation $\langle \rho \rho \rangle = 0$. In the symplectic fermion model the density would be some linear-combination of Ω and ω resulting again in non-vanishing expectation value but the density-density correlation acquires a logarithmic behaviour

$$\langle \rho(z_1) \rho(z_2) \rangle = A + B \ln |z_{12}|^2.$$

For a detailed comparison with lattice correlators it would be useful to determine the amplitudes of the $\mathcal{SF}[\mathcal{C}_4]$ in finite geometries, *e.g.* a rectangular region or strip with various boundary conditions. The short-distance behaviour of these amplitudes will agree with those on the Riemann sphere calculated here but the global behaviour will be modified by finite-size corrections.

The symplectic fermion model provides definite predictions for polymer correlators going beyond just the set of scaling dimensions. It is hoped that numerical lattice simulations will be able to verify these predictions and distinguish them from the \mathbb{Z}_4 -twisted (ξ, η) system. On the lattice the logarithmic behaviour of the correlators may however be masked by discretisation effects.

Similar considerations apply to dilute polymer or percolation models whose continuum limit is described by a $c = 0$ conformal field theory. The scaling dimensions for dilute polymers can be reproduced by the Kac formula of conformal weights for $c = 0$ if one also allows half-integer indices [1]. A conformal field theory with central charge $c = 0$ containing Virasoro primary of these conformal weights cannot be the Virasoro minimal model at $c = 0$ since this has empty field content. By an analysis of fusion products analogous to that described in [13] one can show that a $c = 0$ model which contains fields from the Kac table has to include logarithmic fields. Preliminary results indicate that the generalised highest weight representations occurring have a considerably more complicated structure than in the $c = -2$ case. Further work in this direction will be reported elsewhere.

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A Locality of symplectic fermions

To construct non-chiral local representations we proceed as described in [27]. We start off with the tensor product $\mathcal{A}^\sharp \otimes \bar{\mathcal{A}}^\sharp$ of the left and right chiral representations. In a local theory the operator $S = L_0^{(n)} - \bar{L}_0^{(n)}$, *i.e.* the nilpotent part of $L_0 - \bar{L}_0$, has to vanish on all states. The (maximal) non-chiral representation \mathcal{W}_{\max} is thus given as the quotient space

$$\mathcal{W}_{\max} = (\mathcal{A}^\sharp \otimes \bar{\mathcal{A}}^\sharp) / \mathcal{N}, \quad (66)$$

where \mathcal{N} is the subrepresentation generated from $S(\omega \otimes \bar{\omega})$. The space of ground states then has the structure

$$\begin{aligned} \chi_0^\alpha \omega &= -\theta^\alpha, & \bar{\chi}_0^{\bar{\alpha}} \omega &= -\bar{\theta}^{\bar{\alpha}}, \\ \chi_0^\alpha \theta^\beta &= d^{\alpha\beta} \Omega, & \bar{\chi}_0^{\bar{\alpha}} \bar{\theta}^{\bar{\beta}} &= d^{\alpha\beta} \Omega, \\ \chi_0^\alpha \bar{\theta}^{\bar{\alpha}} &= -\xi^{\alpha\bar{\alpha}}, & \bar{\chi}_0^{\bar{\alpha}} \theta^\alpha &= \xi^{\alpha\bar{\alpha}}. \end{aligned} \quad (67)$$

Here, ω is the equivalence class of states in \mathcal{W}_{\max} which contains $\omega \otimes \bar{\omega}$ as a representative. Since S commutes with both chiral algebras and \mathcal{A}^\sharp is freely generated from the ground space representation by the negative modes $\chi_m^\alpha, \bar{\chi}_m^{\bar{\alpha}}$ with $m < 0$, the same is true of \mathcal{W}_{\max} .

The representation \mathcal{W}_{\max} contains the states which are allowed *a priori* in a local theory. However, the space of states \mathcal{W} actually realised in the non-chiral local theory may be smaller than \mathcal{W}_{\max} , as we might be forced to set some of those states to zero when requiring the locality of two- and three-point amplitudes. We construct these amplitudes by the method described in [27]: The N -point amplitudes are co-invariants with respect to the comultiplications $\Delta^i(\phi_{-n})$, where ϕ is any field in the chiral algebra, $n > -h$ with h the conformal weight of ϕ and $i = 1, \dots, N$. In our case it is sufficient to use the comultiplications

$$\Delta^i(\chi_{-n}^\alpha) = \chi_{-n}^{\alpha(i)} + \sum_{j \neq i} \varepsilon_{ij} \sum_{k=0}^{\infty} \binom{-n}{k} z_{ji}^{-n-k} \chi_k^{\alpha(j)}, \quad (68)$$

where $\chi_k^{\alpha(j)}$ is the mode χ_k^α acting on the j -th field in the amplitude and ε_{ij} is a sign factor arising from interchanging of fermions when moving χ^α from i -th to j -th position. The comultiplication allows us to express any amplitude in terms of amplitudes of the ground states alone. These in turn satisfy systems of first order differential equations obtained by identifying d/dz_i with L_{-1} acting on the i -th field and noting that

$$L_{-1}\Omega = 0, \quad L_{-1}\omega = -d_{\alpha\beta}\chi_{-1}^\alpha\theta^\beta, \quad L_{-1}\theta^\alpha = \chi_{-1}^\alpha\Omega. \quad (69)$$

In this way the bosonic two-point amplitudes are found as

$$\begin{aligned} \langle \omega\omega \rangle &= -2\mathcal{C}_1 - 2\mathcal{C}_0 \ln |z_{12}|^2, \\ \langle \omega\Omega \rangle &= \mathcal{C}_0, \\ \langle \omega\xi^{\alpha\bar{\alpha}} \rangle &= -\mathcal{C}_0\Theta^{\alpha\bar{\alpha}}, \\ \langle \Omega\Omega \rangle &= 0, \\ \langle \Omega\xi^{\alpha\bar{\alpha}} \rangle &= 0, \\ \langle \xi^{\alpha\bar{\alpha}}\xi^{\beta\bar{\beta}} \rangle &= 0, \end{aligned} \quad (70)$$

while the amplitudes of two fermionic fields are

$$\begin{aligned} \langle \theta^\alpha\theta^\beta \rangle &= d^{\alpha\beta}\mathcal{C}_0, \\ \langle \bar{\theta}^{\bar{\alpha}}\bar{\theta}^{\bar{\beta}} \rangle &= d^{\bar{\alpha}\bar{\beta}}\mathcal{C}_0, \\ \langle \theta^\alpha\bar{\theta}^{\bar{\alpha}} \rangle &= -\Theta^{\alpha\bar{\alpha}}\mathcal{C}_0. \end{aligned} \quad (71)$$

Here, $\mathcal{C}_0, \mathcal{C}_1$ and $\Theta^{\alpha\bar{\alpha}}$ are arbitrary constants. Amplitudes of one fermionic and one bosonic field vanish. These amplitudes imply that $\xi^{\alpha\bar{\alpha}}$ and Ω are linearly dependent and thus¹

$$\xi^{\alpha\bar{\alpha}} = -\Theta^{\alpha\bar{\alpha}}\Omega. \quad (72)$$

¹We set states to zero whose amplitudes vanish identically.

The fermionic states θ^α and $\bar{\theta}^{\bar{\alpha}}$ are linearly dependent provided $\det(\Theta) = 1$. This is in fact required by locality of the amplitude $\langle \theta^\alpha \theta^\beta \omega \rangle$: Solving the system of differential equations arising from the comultiplication we obtain

$$\langle \theta^\alpha \theta^\beta \omega \rangle = -d^{\alpha\beta} \left(\mathcal{A}_1 + \mathcal{A}_0 \ln \frac{z_{13} z_{23}}{z_{12}} + \det(\Theta) \mathcal{A}_0 \ln \frac{\bar{z}_{13} \bar{z}_{23}}{\bar{z}_{12}} \right). \quad (73)$$

This amplitude can be local only if $\det(\Theta) = 1$. Then the four fermionic states θ^α and $\bar{\theta}^{\bar{\alpha}}$ are related as

$$\theta^\alpha = \Theta^{\alpha\bar{\alpha}} d_{\bar{\alpha}\beta} \bar{\theta}^{\bar{\beta}}, \quad \bar{\theta}^{\bar{\alpha}} = -\Theta^{\alpha\bar{\alpha}} d_{\alpha\beta} \theta^\beta. \quad (74)$$

The space of states \mathcal{W} is then freely generated by the negative modes, χ_m^α and $\bar{\chi}_m^\alpha$ with $m < 0$, from the four ground states; two bosonic states, Ω and ω , and two fermionic states θ^α

We now determine the remaining three-point amplitudes.

$$\begin{aligned} \langle \Omega \Omega \Omega \rangle &= 0, \\ \langle \omega \Omega \Omega \rangle &= \mathcal{A}_0, \\ \langle \omega \omega \Omega \rangle &= -2\mathcal{A}_1 - 2\mathcal{A}_0 \ln |z_{12}|^2, \\ \langle \omega \omega \omega \rangle &= 3\mathcal{A}_2 + 2\mathcal{A}_1 \ln |z_{12} z_{13} z_{23}|^2 \\ &\quad + 2\mathcal{A}_0 (\ln |z_{12}|^2 \ln |z_{13}|^2 + \ln |z_{12}|^2 \ln |z_{23}|^2 + \ln |z_{13}|^2 \ln |z_{23}|^2) \\ &\quad - \mathcal{A}_0 ((\ln |z_{12}|^2)^2 + (\ln |z_{13}|^2)^2 + (\ln |z_{23}|^2)^2), \\ \langle \theta^\alpha \theta^\beta \Omega \rangle &= d^{\alpha\beta} \mathcal{A}_0, \\ \langle \theta^\alpha \theta^\beta \omega \rangle &= -d^{\alpha\beta} \left(\mathcal{A}_1 + \mathcal{A}_0 \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right). \end{aligned} \quad (75)$$

From the two- and three-point amplitudes we can read off the OPEs. Locality requires that the different ways of contracting two fields in a three-point amplitude are equivalent; this implies

$$\frac{\mathcal{C}_1}{\mathcal{C}_0} = \frac{\mathcal{A}_1}{\mathcal{A}_0} = \frac{\mathcal{A}_2}{\mathcal{A}_1} = \mathcal{Z}. \quad (76)$$

We further introduce parameters Λ and \mathcal{O} by $\mathcal{C}_0 = \Lambda^4 \mathcal{O}$, $\mathcal{A}_0 = \Lambda^6 \mathcal{O}$. The OPEs are then

$$\begin{aligned} \Lambda^{-2} \theta^\alpha(x) \theta^\beta &= d^{\alpha\beta} (\omega + (\mathcal{Z} + \ln |x|^2) \Omega), \\ \Lambda^{-2} \theta^\alpha(x) \omega &= -(\mathcal{Z} + \ln |x|^2) \theta^\alpha, \\ \Lambda^{-2} \omega(x) \omega &= -(\mathcal{Z} + \ln |x|^2) (2\omega + (\mathcal{Z} + \ln |x|^2) \Omega), \end{aligned} \quad (77)$$

The operator product of Ω with any field \mathbf{S} is simply given by $\Omega(x) \mathbf{S} = \Lambda^2 \mathbf{S}$ to all orders. For $\Lambda^2 = 1$, the fields Ω can be thought as the unit operator, except that its one-point function vanishes, $\langle \Omega \rangle = 0$.

The two- and three-point amplitudes completely determine the theory and all higher amplitudes can be constructed, at least in principle, from these by a gluing process. Conversely, by taking limits of n -point amplitudes in which two fields are close together,

we can relate n -point amplitudes to $(n-1)$ -point amplitudes and the OPEs. The different ways of so contracting two fields all have to be compatible and this leads to consistency relations. It is sufficient to check these for the four-point amplitudes. These are listed in the main part of the paper and do indeed satisfy all consistency relations. This shows that the symplectic fermions define a local conformal field theory.

In the main part of this paper we revert to regular greek letters instead of bold ones to denote the non-chiral fields. We also fix the parameters as $\Lambda = \Omega = 1$. They could easily be restored by multiplying all n -point amplitudes by $\Lambda^{2n}\mathcal{O}$. Furthermore, by performing a global chiral (or anti-chiral) $SU(2)$ transformation, we can choose $\Theta^{\alpha\bar{\alpha}} = d^{\alpha\bar{\alpha}}$. The only remaining parameter, \mathcal{Z} , corresponds to the freedom of adding to ω an arbitrary multiple of Ω .

B Twisted amplitudes

In abelian orbifolds the twist conditions satisfied by the twist fields in any amplitude all commute. This implies that the twists can be simultaneously diagonalised and insertion points of twist fields are rational branch points for the symplectic fermions. To calculate amplitudes involving twisted fields we can then use a twisted comultiplication for the chiral fermion fields. To derive the comultiplication formula consider (non-chiral) fields ϕ_j having twists α_j with respect to a chiral field $S(w)$. If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in \mathbb{Z}$ then, as a function of w ,

$$(w - z_1)^{-\alpha_1}(w - z_2)^{-\alpha_2}(w - z_3)^{-\alpha_3}(w - z_4)^{-\alpha_4}\langle S(w)\phi_1\phi_2\phi_3\phi_4\rangle, \quad (78)$$

is meromorphic with poles at $w = z_j$. Taking a contour integral of w around the other fields we obtain the comultiplication formula for S ,

$$\begin{aligned} \Delta(S_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-h_S+1}) = & \\ & \sum_{r,s,t=0}^{\infty} \binom{-\alpha_2}{r} \binom{-\alpha_3}{s} \binom{-\alpha_4}{t} z_{12}^{-\alpha_2-r} z_{13}^{-\alpha_3-s} z_{14}^{-\alpha_4-t} S_{-\alpha_1+r+s+t-h_S+1}^{(1)} \\ & + \sum_{r,s,t=0}^{\infty} \binom{-\alpha_1}{r} \binom{-\alpha_3}{s} \binom{-\alpha_4}{t} z_{21}^{-\alpha_1-r} z_{23}^{-\alpha_3-s} z_{24}^{-\alpha_4-t} S_{-\alpha_2+r+s+t-h_S+1}^{(2)} \\ & + \sum_{r,s,t=0}^{\infty} \binom{-\alpha_1}{r} \binom{-\alpha_2}{s} \binom{-\alpha_4}{t} z_{31}^{-\alpha_1-r} z_{32}^{-\alpha_2-s} z_{34}^{-\alpha_4-t} S_{-\alpha_3+r+s+t-h_S+1}^{(3)} \\ & + \sum_{r,s,t=0}^{\infty} \binom{-\alpha_1}{r} \binom{-\alpha_2}{s} \binom{-\alpha_3}{t} z_{41}^{-\alpha_1-r} z_{42}^{-\alpha_2-s} z_{43}^{-\alpha_3-t} S_{-\alpha_4+r+s+t-h_S+1}^{(4)}, \end{aligned} \quad (79)$$

where $S_m^{(j)}$ acts on the j th field. If some fields are fermionic there are additional minus signs from the interchange of two fermion fields. As in the untwisted case any amplitude functional Φ satisfies

$$\Phi \circ \Delta(S_m) = 0 \quad \text{for } m < h_S, \quad (80)$$

that is, $\sum \alpha_j \geq -2(h_S - 1)$. Two- and three-point amplitudes satisfy analogous comultiplication properties.

We can use the comultiplication to derive differential equations for the amplitudes. Note that

$$\begin{aligned} L_{-1}\mu &= \chi_{\lambda-1}^- \chi_{-\lambda}^+ \mu, \\ L_{-1}^2 \mu &= (1-\lambda) \chi_{\lambda-2}^- \chi_{-\lambda}^+ \mu - \lambda \chi_{-1-\lambda}^+ \chi_{\lambda-1}^- \mu. \end{aligned} \quad (81)$$

To derive a differential equation for $\langle \mu_{\lambda_1} \cdots \rangle$ we typically use the comultiplication for χ^- with $\alpha_1 = 1 - \lambda_1$ and all other $\alpha_j \leq 0$ such that $\sum_j \alpha_j = 0$ to change the $\chi_{\lambda-1}^-$ mode on the first field into χ_0^- acting on the fields with $\alpha_j = 0$. A second application of the comultiplication for χ^+ with $\alpha_1 = \lambda_1$ and all other $\alpha_j \leq 0$ such that $\sum_j \alpha_j = 0$ then changes the $\chi_{-\lambda}^+$ mode on the first field into χ_+^+ acting on the fields with $\alpha_j = 0$. In cases where we cannot satisfy these conditions on the α_j we can derive a second order differential equation instead, as detailed below.

B.1 Two twist fields

Amplitudes with two twist fields, $\langle \mu_{\lambda_1} \mu_{\lambda_2} \cdots \rangle$ are non-zero only if $\lambda_1 + \lambda_2 = 1$. They satisfy first order differential equations, for example,

$$\left(\partial_1 - \frac{\lambda \lambda^*}{z_{12}} \right) \langle \mu_\lambda \mu_{\lambda^*} \rangle = 0, \quad (82)$$

$$\left(\partial_1 - \frac{\lambda \lambda^*}{z_{12}} \right) \langle \mu_\lambda \mu_{\lambda^*} \omega \rangle = \frac{z_{23}}{z_{12} z_{13}} \langle \mu_\lambda \mu_{\lambda^*} \Omega \rangle, \quad (83)$$

$$\left(\partial_1 - \frac{\lambda \lambda^*}{z_{12}} \right) \langle \mu_\lambda \mu_{\lambda^*} \theta^+ \theta^- \rangle = \frac{z_{23}}{z_{12} z_{13}} \left(\frac{z_{13} z_{24}}{z_{14} z_{23}} \right)^\lambda \langle \mu_\lambda \mu_{\lambda^*} \Omega \Omega \rangle, \quad (84)$$

$$\begin{aligned} \left(\partial_1 - \frac{\lambda \lambda^*}{z_{12}} \right) \langle \mu_\lambda \mu_{\lambda^*} \omega \omega \rangle &= \frac{z_{23}}{z_{12} z_{13}} \langle \mu_\lambda \mu_{\lambda^*} \Omega \omega \rangle + \frac{z_{24}}{z_{12} z_{14}} \langle \mu_\lambda \mu_{\lambda^*} \omega \Omega \rangle \\ &\quad + \frac{z_{23}}{z_{12} z_{13}} \left(\frac{z_{13} z_{24}}{z_{14} z_{23}} \right)^\lambda \langle \mu_\lambda \mu_{\lambda^*} \theta^- \theta^+ \rangle \\ &\quad - \frac{z_{24}}{z_{12} z_{14}} \left(\frac{z_{14} z_{23}}{z_{13} z_{24}} \right)^\lambda \langle \mu_\lambda \mu_{\lambda^*} \theta^+ \theta^- \rangle. \end{aligned} \quad (85)$$

Imposing Möbius covariance and monodromy invariance fixes the functional form of the amplitudes,

$$\langle \mu_\lambda \mu_{\lambda^*} \rangle = \mathcal{D}_\lambda |z_{12}|^{2\lambda\lambda^*}, \quad (86)$$

$$\langle \mu_\lambda \mu_{\lambda^*} \Omega \rangle = \mathcal{D}_\lambda^{(1)} |z_{12}|^{2\lambda\lambda^*}, \quad (87)$$

$$\langle \mu_\lambda \mu_{\lambda^*} \omega \rangle = -\mathcal{D}_\lambda^{(1)} |z_{12}|^{2\lambda\lambda^*} \left(\mathcal{Z}_\lambda + \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right), \quad (88)$$

$$\langle \mu_\lambda \mu_{\lambda^*} \Omega \Omega \rangle = \mathcal{D}_\lambda^{(2)} |z_{12}|^{2\lambda\lambda^*}, \quad (89)$$

$$\langle \mu_\lambda \mu_{\lambda^*} \omega \Omega \rangle = -\mathcal{D}_\lambda^{(2)} |z_{12}|^{2\lambda\lambda^*} \left(\mathcal{Z}_\lambda^{(1)} + \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right), \quad (90)$$

$$\langle \mu_\lambda \mu_{\lambda^*} \Omega \omega \rangle = -\mathcal{D}_\lambda^{(2)} |z_{12}|^{2\lambda\lambda^*} \left(\mathcal{Z}_\lambda^{(2)} + \ln \left| \frac{z_{14} z_{24}}{z_{12}} \right|^2 \right), \quad (91)$$

$$\langle \mu_\lambda \mu_{\lambda^*} \theta^+ \theta^- \rangle = \mathcal{D}_\lambda^{(2)} |z_{12}|^{2\lambda\lambda^*} H^\lambda(x, \bar{x}), \quad (92)$$

$$\langle \mu_\lambda \mu_{\lambda^*} \theta^- \theta^+ \rangle = -\mathcal{D}_\lambda^{(2)} |z_{12}|^{2\lambda\lambda^*} H^{\lambda^*}(x, \bar{x}), \quad (93)$$

$$\begin{aligned} \langle \mu_\lambda \mu_{\lambda^*} \omega \omega \rangle &= \mathcal{D}_\lambda^{(2)} |z_{12}|^{2\lambda\lambda^*} \left[\left(\mathcal{Z}_\lambda^{(1)} + \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right) \left(\mathcal{Z}_\lambda^{(2)} + \ln \left| \frac{z_{14} z_{24}}{z_{12}} \right|^2 \right) \right. \\ &\quad \left. - H^\lambda(x, \bar{x}) H^{\lambda^*}(x, \bar{x}) + \mathcal{X}_\lambda \right], \end{aligned} \quad (94)$$

where $H^\lambda(x, \bar{x})$ is given by

$$\begin{aligned} H^\lambda(x, \bar{x}) &= -2\Re \left(\frac{(1-x)^\lambda}{\lambda} {}_2F_1(1, \lambda; 1+\lambda; 1-x) \right) + \mathcal{Y}_\lambda \\ &= \ln |x|^2 - 2\vartheta_{1,\lambda} + \mathcal{Y}_\lambda + 2\Re \left((1-x)^\lambda M(1, \lambda; 1; x) \right) \end{aligned} \quad (95)$$

Reading off the OPEs from the two- and three-point amplitudes and imposing locality on the four-point amplitudes fixes the constants appearing in the amplitudes in terms of two new free parameters, \mathcal{O}_λ and \mathcal{Z}_λ , for each pair of conjugate twisted sectors,

$$\begin{aligned} \mathcal{D}_\lambda &= -\Lambda^2 \mathcal{O}_\lambda, & \mathcal{D}_\lambda^{(1)} &= -\Lambda^4 \mathcal{O}_\lambda, & \mathcal{D}_\lambda^{(2)} &= -\Lambda^6 \mathcal{O}_\lambda, \\ \mathcal{Z}_\lambda^{(1)} &= \mathcal{Z}_\lambda^{(2)} = \mathcal{Z}_\lambda, & \mathcal{Y}_\lambda &= \mathcal{Z} - \mathcal{Z}_\lambda + 2\vartheta_{1,\lambda}, & \mathcal{X}_\lambda &= 0. \end{aligned} \quad (96)$$

The OPEs are then given by

$$\mu_\lambda(x) \mu_{1-\lambda} = -\frac{\mathcal{O}_\lambda}{\mathcal{O}} |x|^{2\lambda(1-\lambda)} \left(\omega + (\ln |x|^2 + 2\mathcal{Z} - \mathcal{Z}_\lambda) \Omega \right), \quad (97)$$

$$\Lambda^{-2} \mu_\lambda(x) \Omega = \mu_\lambda, \quad (98)$$

$$\Lambda^{-2} \mu_\lambda(x) \omega = -\left(\ln |x|^2 + \mathcal{Z}_\lambda \right) \mu_\lambda. \quad (99)$$

B.2 Three twist fields

For the three-point amplitude $\langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \rangle$ we have to consider two cases. If $\lambda_1 + \lambda_2 + \lambda_3 = 1$ the amplitude satisfies

$$\left[\partial_1 - \lambda_1 \left(\frac{\lambda_2}{z_{12}} + \frac{\lambda_3}{z_{13}} \right) \right] \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \rangle = 0. \quad (100)$$

while for $\lambda_1 + \lambda_2 + \lambda_3 = 2$, we have

$$\left[\partial_1 - (1 - \lambda_1) \left(\frac{1 - \lambda_2}{z_{12}} + \frac{1 - \lambda_3}{z_{13}} \right) \right] \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \rangle = 0. \quad (101)$$

From this we obtain the amplitudes

$$\begin{aligned}\langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \rangle &= \mathcal{C}_{\lambda_1, \lambda_2, \lambda_3} \left| z_{12}^{\lambda_1 \lambda_2} z_{13}^{\lambda_1 \lambda_3} z_{23}^{\lambda_2 \lambda_3} \right|^2 \quad \text{for } \lambda_1 + \lambda_2 + \lambda_3 = 1, \\ &= \mathcal{C}_{\lambda_1, \lambda_2, \lambda_3} \left| z_{12}^{\lambda_1^* \lambda_2^*} z_{13}^{\lambda_1^* \lambda_3^*} z_{23}^{\lambda_2^* \lambda_3^*} \right|^2 \quad \text{for } \lambda_1 + \lambda_2 + \lambda_3 = 2,\end{aligned}\tag{102}$$

B.3 Four twist fields

In the case of four twist fields we obtain

$$\left[\partial_1 - \lambda_1 \left(\frac{\lambda_2}{z_{12}} + \frac{\lambda_3}{z_{13}} + \frac{\lambda_4}{z_{14}} \right) \right] \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle = 0.\tag{103}$$

for $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ and

$$\left[\partial_1 - (1 - \lambda_1) \left(\frac{1 - \lambda_2}{z_{12}} + \frac{1 - \lambda_3}{z_{13}} + \frac{1 - \lambda_4}{z_{14}} \right) \right] \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle = 0.\tag{104}$$

for $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 3$. The four-point amplitudes, manifestly symmetric in the fields, are thus

$$\begin{aligned}\langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle &= \mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \left| \prod_{i < j} z_{ij}^{\lambda_i \lambda_j} \right|^2 \quad \text{for } \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\ &= \mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \left| \prod_{i < j} z_{ij}^{\lambda_i^* \lambda_j^*} \right|^2 \quad \text{for } \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 3.\end{aligned}\tag{105}$$

The last case, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2$, requires a second order differential equation. Using $\alpha_1 = 2 - \lambda_1, \alpha_2 = -\lambda_2, \alpha_3 = -\lambda_3, \alpha_4 = -\lambda_4$ and $\alpha_1 = 1 + \lambda_1, \alpha_2 = \lambda_2 - 1, \alpha_3 = \lambda_3 - 1, \alpha_4 = \lambda_4 - 1$ we obtain

$$\begin{aligned}&\left\{ \partial_1^2 + \left[\lambda_1^* \left(\frac{\lambda_2}{z_{12}} + \frac{\lambda_3}{z_{13}} + \frac{\lambda_4}{z_{14}} \right) + \lambda_1 \left(\frac{\lambda_2^*}{z_{12}} + \frac{\lambda_3^*}{z_{13}} + \frac{\lambda_4^*}{z_{14}} \right) \right] \partial_1 \right. \\ &\left. + \lambda_1 \lambda_1^* \left(\frac{\lambda_2 \lambda_2^*}{z_{12}^2} + \frac{\lambda_3 \lambda_3^*}{z_{13}^2} + \frac{\lambda_4 \lambda_4^*}{z_{14}^2} - \frac{\lambda_2 \lambda_3 + \lambda_2^* \lambda_3^*}{z_{12} z_{13}} - \frac{\lambda_2 \lambda_4 + \lambda_2^* \lambda_4^*}{z_{12} z_{14}} - \frac{\lambda_3 \lambda_4 + \lambda_3^* \lambda_4^*}{z_{13} z_{14}} \right) \right\} \\ &\times \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle = 0\end{aligned}$$

Möbius covariance implies the amplitude can be written as

$$\begin{aligned}\langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle &= \\ &= \left| z_{12}^{-h_1 - h_2 + h_3 + h_4} z_{13}^{-2h_3} z_{14}^{-h_1 + h_2 + h_3 - h_4} z_{24}^{h_1 - h_2 - h_3 - h_4} \right|^2 \left| x^{\lambda_3 \lambda_4} (1 - x)^{\lambda_2 \lambda_3} \right|^2 f(x, \bar{x}), \\ &= \left| z_{12}^{\lambda_1^* \lambda_2^*} z_{34}^{\lambda_3 \lambda_4} z_{14}^{\lambda_1^* \lambda_4^*} z_{23}^{\lambda_2 \lambda_3} z_{13}^{\lambda_1^* \lambda_3^*} z_{24}^{\lambda_2 \lambda_4} (z_{13} z_{24})^{-\lambda_1^*} \right|^2 f(x, \bar{x}),\end{aligned}\tag{106}$$

where $f(x, \bar{x})$ satisfies the hypergeometric equation of type $(\lambda_3, 1 - \lambda_1, \lambda_3 + \lambda_4)$,

$$x(1 - x)f'' + [\lambda_3 + \lambda_4 - (\lambda_2 + 2\lambda_3 + \lambda_4)x]f' - \lambda_3(\lambda_2 + \lambda_3 + \lambda_4 - 1)f = 0.$$

If no two λ_j add up to an integer, a system of two linear independent solutions in the vicinity of $x = 0$ is given by

$$\begin{aligned} f_1^{(0)} &= {}_2F_1(\lambda_3, 1 - \lambda_1; \lambda_3 + \lambda_4; x), \\ f_2^{(0)} &= x^{1-\lambda_3-\lambda_4}(1-x)^{1-\lambda_2-\lambda_3} {}_2F_1(\lambda_1, 1 - \lambda_3; \lambda_1 + \lambda_2; x). \end{aligned} \quad (107)$$

Another set of solutions, adapted to $x \sim 1$, is

$$\begin{aligned} f_1^{(1)} &= {}_2F_1(\lambda_3, 1 - \lambda_1; \lambda_2 + \lambda_3; 1 - x), \\ f_2^{(1)} &= x^{1-\lambda_3-\lambda_4}(1-x)^{1-\lambda_2-\lambda_3} {}_2F_1(\lambda_1, 1 - \lambda_3; \lambda_1 + \lambda_4; 1 - x), \end{aligned} \quad (108)$$

while a set of solutions in the vicinity of $x = \infty$ is given by

$$\begin{aligned} f_1^{(\infty)} &= x^{-\lambda_3} {}_2F_1(\lambda_3, 1 - \lambda_4; \lambda_1 + \lambda_3; 1/x), \\ f_2^{(\infty)} &= x^{-\lambda_4}(1-x)^{1-\lambda_2-\lambda_3} {}_2F_1(\lambda_4, 1 - \lambda_3; \lambda_2 + \lambda_4; 1/x). \end{aligned} \quad (109)$$

Monodromy invariance requires that the non-chiral solution is of the form, up to an overall constant,

$$\begin{aligned} f(x, \bar{x}) &= -\rho(\lambda_3, \lambda_4)|f_1^{(0)}|^2 - \rho(\lambda_1, \lambda_2)|f_2^{(0)}|^2 \\ &= -\rho(\lambda_2, \lambda_3)|f_1^{(1)}|^2 - \rho(\lambda_1, \lambda_4)|f_2^{(1)}|^2 \\ &= -\rho(\lambda_1, \lambda_3)|f_1^{(\infty)}|^2 - \rho(\lambda_2, \lambda_4)|f_2^{(\infty)}|^2, \end{aligned}$$

where

$$\rho(\lambda, \lambda') = \frac{\Gamma(1 - \lambda - \lambda')}{\Gamma(\lambda + \lambda')} \frac{\Gamma(\lambda)}{\Gamma(1 - \lambda)} \frac{\Gamma(\lambda')}{\Gamma(1 - \lambda')} = \rho(\lambda, 1 - \lambda - \lambda').$$

Note that, if $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2$ we have

$$\frac{\rho(1 - \lambda_1, 1 - \lambda_2)}{\rho(\lambda_3, \lambda_4)} = \frac{\rho(1 - \lambda_3, 1 - \lambda_4)}{\rho(\lambda_1, \lambda_2)} = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)}{\Gamma(1 - \lambda_1)\Gamma(1 - \lambda_2)\Gamma(1 - \lambda_3)\Gamma(1 - \lambda_4)}.$$

Thus the full four-point amplitude, manifestly symmetric in the fields, can be written as

$$\begin{aligned} \mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}^{-1} \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle &= \\ &= -\sqrt{\rho(\lambda_3, \lambda_4)\rho(\lambda_1^*, \lambda_2^*)} \left| z_{12}^{\lambda_1^* \lambda_2^*} z_{34}^{\lambda_3 \lambda_4} z_{14}^{\lambda_1^* \lambda_4^*} z_{23}^{\lambda_2 \lambda_3} z_{13}^{\lambda_1^* \lambda_3^*} z_{24}^{\lambda_2 \lambda_4} (z_{13} z_{24})^{-\lambda_1^*} \right|^2 \\ &\quad \times |{}_2F_1(\lambda_3, 1 - \lambda_1; \lambda_3 + \lambda_4; x)|^2 \\ &\quad - \sqrt{\rho(\lambda_1, \lambda_2)\rho(\lambda_3^*, \lambda_4^*)} \left| z_{12}^{\lambda_1 \lambda_2} z_{34}^{\lambda_3^* \lambda_4^*} z_{14}^{\lambda_1 \lambda_4} z_{23}^{\lambda_2^* \lambda_3^*} z_{13}^{\lambda_1 \lambda_3} z_{24}^{\lambda_2^* \lambda_4^*} (z_{13} z_{24})^{-\lambda_3^*} \right|^2 \\ &\quad \times |{}_2F_1(\lambda_1, 1 - \lambda_3; \lambda_1 + \lambda_2; x)|^2 \end{aligned} \quad (110)$$

$$\begin{aligned}
&= -\sqrt{\rho(\lambda_2, \lambda_3)\rho(\lambda_1^*, \lambda_4^*)} \left| z_{12}^{\lambda_1^* \lambda_2^*} z_{34}^{\lambda_3 \lambda_4} z_{14}^{\lambda_1^* \lambda_4^*} z_{23}^{\lambda_2 \lambda_3} z_{13}^{\lambda_1^* \lambda_3^*} z_{24}^{\lambda_2 \lambda_4} (z_{13} z_{24})^{-\lambda_1^*} \right|^2 \\
&\quad \times |{}_2F_1(\lambda_3, 1 - \lambda_1; \lambda_2 + \lambda_3; 1 - x)|^2 \\
&\quad - \sqrt{\rho(\lambda_1, \lambda_4)\rho(\lambda_2^*, \lambda_3^*)} \left| z_{12}^{\lambda_1 \lambda_2} z_{34}^{\lambda_3^* \lambda_4^*} z_{14}^{\lambda_1 \lambda_4} z_{23}^{\lambda_2^* \lambda_3^*} z_{13}^{\lambda_1 \lambda_3} z_{24}^{\lambda_2^* \lambda_4^*} (z_{13} z_{24})^{-\lambda_3^*} \right|^2 \\
&\quad \times |{}_2F_1(\lambda_1, 1 - \lambda_3; \lambda_1 + \lambda_4; 1 - x)|^2 \tag{111}
\end{aligned}$$

$$\begin{aligned}
&= -\sqrt{\rho(\lambda_1, \lambda_3)\rho(\lambda_1^*, \lambda_4^*)} \left| z_{12}^{\lambda_1 \lambda_2} z_{34}^{\lambda_3^* \lambda_4^*} z_{14}^{\lambda_1^* \lambda_4^*} z_{23}^{\lambda_2 \lambda_3} z_{13}^{\lambda_1 \lambda_3} z_{24}^{\lambda_2^* \lambda_4^*} (z_{12} z_{34})^{-\lambda_4^*} \right|^2 \\
&\quad \times |{}_2F_1(\lambda_3, 1 - \lambda_4; \lambda_1 + \lambda_3; 1/x)|^2 \\
&\quad - \sqrt{\rho(\lambda_1, \lambda_4)\rho(\lambda_1^*, \lambda_3^*)} \left| z_{12}^{\lambda_1 \lambda_2} z_{34}^{\lambda_3^* \lambda_4^*} z_{14}^{\lambda_1 \lambda_4} z_{23}^{\lambda_2^* \lambda_3^*} z_{13}^{\lambda_1^* \lambda_3^*} z_{24}^{\lambda_2 \lambda_4} (z_{13} z_{24})^{-\lambda_3^*} \right|^2 \\
&\quad \times |{}_2F_1(\lambda_4, 1 - \lambda_3; \lambda_2 + \lambda_4; 1/x)|^2 \tag{112}
\end{aligned}$$

in the vicinity of $x \sim 0, 1$ and ∞ , respectively. The constant $\mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}$ is independent of the ordering of the indices.

If two λ_j add up to an integer, $\lambda_1 + \lambda_4 = 1, \lambda_2 + \lambda_3 = 1, \lambda_3 + \lambda_4 \neq 1$ say, we encounter a degenerate case of the hypergeometric equation. We still have the same system of solutions near $x = 0$,

$$\begin{aligned}
f_1^{(0)} &= {}_2F_1(\lambda_3, \lambda_4; \lambda_3 + \lambda_4; x), \\
f_2^{(0)} &= x^{1-\lambda_3-\lambda_4} {}_2F_1(1 - \lambda_3, 1 - \lambda_4; 2 - \lambda_3 - \lambda_4; x).
\end{aligned}$$

Their analytic continuation to $x \sim 1$ is given by

$$\begin{aligned}
f_1^{(0)} &= -\frac{\Gamma(\lambda_3 + \lambda_4)}{\Gamma(\lambda_3)\Gamma(\lambda_4)} \left(f^{(1)}[\ln(1 - x) - \vartheta_{\lambda_3, \lambda_4}] + \tilde{f}^{(1)} \right), \\
f_2^{(0)} &= -\frac{\Gamma(2 - \lambda_3 - \lambda_4)}{\Gamma(1 - \lambda_3)\Gamma(1 - \lambda_4)} \left(f^{(1)}[\ln(1 - x) - \vartheta_{1-\lambda_3, 1-\lambda_4}] + \tilde{f}^{(1)} \right),
\end{aligned}$$

where

$$\begin{aligned}
f_1 &= {}_2F_1(\lambda_3, \lambda_4; 1; 1 - x), \\
\tilde{f}_1 &= M(\lambda_3, \lambda_4; 1; 1 - x), \\
\vartheta_{a,b} &= 2\psi(1) - \psi(a) - \psi(b), \\
M(a, b; c; x) &= \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} [h_{a,n} + h_{b,n} - h_{c,n} - h_{1,n}] x^n, \\
h_{a,n} &= \psi(a + n) - \psi(a),
\end{aligned}$$

and $\psi(x) = \Gamma(x)'/\Gamma(x)$ is the digamma function. The monodromy invariant solution is

$$\begin{aligned}
f(x, \bar{x}) &= -\rho(\lambda_3, \lambda_4)|f_1^{(0)}|^2 - \rho(\lambda_1, \lambda_2)|f_2^{(0)}|^2 \\
&= |{}_2F_1(\lambda_3, \lambda_4; 1; 1 - x)|^2 (\ln|1 - x|^2 - \vartheta_{\lambda_3, 1-\lambda_3} - \vartheta_{\lambda_4, 1-\lambda_4}) \\
&\quad + ({}_2F_1(\lambda_3, \lambda_4; 1; 1 - x)M(\lambda_3, \lambda_4; 1; 1 - \bar{x}) + \text{c.c.}) .
\end{aligned}$$

If $\lambda_3 \neq \lambda_4$, that is we have two different pairs of conjugate fields, the solution near $x \sim \infty$ is still given by (112). If $\lambda_3 = \lambda_4 = \lambda$ then

$$f(x, \bar{x}) = |x|^{-2\lambda} \left[|{}_2F_1(\lambda, 1-\lambda; 1; 1/x)|^2 (-\ln |x|^2 - 2\vartheta_{\lambda, 1-\lambda}) + ({}_2F_1(\lambda, 1-\lambda; 1; 1/x)M(\lambda, 1-\lambda; 1; 1/\bar{x}) + \text{c.c.}) \right].$$

One can deal similarly with the other cases where we have one or more pairs of conjugate fields. The generic solution for all cases is given by (110) – (112), which are replaced by a degenerate solution in the following cases, respectively:

$$\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = 1:$$

$$\begin{aligned} \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle &= \mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} |z_{12}|^{2\lambda_1 \lambda_2} |z_{34}|^{2\lambda_3 \lambda_4} \left| \frac{z_{14} z_{23}}{z_{13} z_{24}} \right|^{2\lambda_2 \lambda_3} \\ &\times \left[|{}_2F_1(\lambda_2, \lambda_3; 1; x)|^2 (\ln |x|^2 - \vartheta_{\lambda_1, \lambda_2} - \vartheta_{\lambda_3, \lambda_4}) \right. \\ &\left. + ({}_2F_1(\lambda_2, \lambda_3; 1; x)M(\lambda_2, \lambda_3; 1; \bar{x}) + \text{c.c.}) \right] \end{aligned}$$

$$\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3 = 1:$$

$$\begin{aligned} \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle &= \mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} |z_{14}|^{2\lambda_1 \lambda_4} |z_{23}|^{2\lambda_2 \lambda_3} \left| \frac{z_{12} z_{34}}{z_{13} z_{24}} \right|^{2\lambda_3 \lambda_4} \\ &\times \left[|{}_2F_1(\lambda_3, \lambda_4; 1; 1-x)|^2 (\ln |1-x|^2 - \vartheta_{\lambda_1, \lambda_4} - \vartheta_{\lambda_2, \lambda_3}) \right. \\ &\left. + ({}_2F_1(\lambda_3, \lambda_4; 1; 1-x)M(\lambda_3, \lambda_4; 1; 1-\bar{x}) + \text{c.c.}) \right] \end{aligned}$$

$$\lambda_1 + \lambda_3 = \lambda_2 + \lambda_4 = 1:$$

$$\begin{aligned} \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \mu_{\lambda_4} \rangle &= \mathcal{F}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} |z_{13}|^{2\lambda_1 \lambda_3} |z_{24}|^{2\lambda_2 \lambda_4} \left| \frac{z_{14} z_{23}}{z_{12} z_{34}} \right|^{2\lambda_2 \lambda_3} \\ &\times \left[|{}_2F_1(\lambda_2, \lambda_3; 1; 1/x)|^2 (-\ln |x|^2 - \vartheta_{\lambda_1, \lambda_3} - \vartheta_{\lambda_2, \lambda_4}) \right. \\ &\left. + ({}_2F_1(\lambda_2, \lambda_3; 1; 1/x)M(\lambda_2, \lambda_3; 1; 1/\bar{x}) + \text{c.c.}) \right] \end{aligned}$$

C Character formulae

We derive here the characters of \mathcal{A}_μ in the \mathcal{C}_{2N} orbifold model. Realising \mathcal{C}_{2N} as the $2N$ -th roots of unity we can use the Jacobi triple product identity to obtain

$$\chi_{\mathcal{A}}(\tau, u) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 + uq^n)(1 + u^{-1}q^n)$$

$$\begin{aligned}
&= \begin{cases} \sum_{m=0}^{2N-1} \frac{u^m}{1+u^{-1}} \Lambda_{(2m+1)N, 2N^2}(\tau) & \text{for } u \neq -1 \\ \eta(\tau)^2 & \text{for } u = -1 \end{cases} \\
&= \sum_{k=0}^{2N-1} u^k \chi_{\mathcal{A}_{k/2N}}(\tau),
\end{aligned}$$

where $u \in \mathbb{C}$ with $u^{2N} = 1$. The Virasoro characters of the spaces $\mathcal{A}_{k/2N}$ can then be obtained by

$$\begin{aligned}
\chi_{\mathcal{A}_{k/2N}}(\tau) &= \frac{1}{2N} \sum_{l=0}^{2N-1} \omega^{-kl} \chi_{\mathcal{A}}(\tau, \omega^l) \\
&= \frac{1}{2N} \left[(-1)^k \eta(\tau)^2 + \sum_{m=0}^{2N-1} \left(\sum_{l=-N+1}^{N-1} \frac{\omega^{(m-k)l}}{1+\omega^{-l}} \right) \Lambda_{(2m+1)N, 2N^2}(\tau) \right] \\
&= \frac{1}{2N} \left[(-1)^k \eta(\tau)^2 + \sum_{m=0}^{N-1} \left(\sum_{l=-N+1}^{N-1} \omega^{-kl} \frac{\omega^{ml} + \omega^{-(m+1)l}}{1+\omega^{-l}} \right) \Lambda_{(2m+1)N, 2N^2}(\tau) \right]
\end{aligned}$$

where $\omega = \exp(\pi i/N)$ is the primitive $2N$ -th root of unity. The coefficients can be evaluated further as

$$\begin{aligned}
\sum_{l=-N+1}^{N-1} \omega^{-kl} \frac{\omega^{ml} + \omega^{-(m+1)l}}{1+\omega^{-l}} &= \sum_{l=-N+1}^{N-1} \omega^{-kl} \sum_{r=-m}^m (-1)^{r+m} \omega^{rl} \\
&= \sum_{r=-m}^m (-1)^{r+m} \sum_{l=-N+1}^{N-1} \omega^{(r-k)l} \\
&= \sum_{r=-m}^m (-1)^{r+m} (2N\delta_{r,k} - (-1)^{r-k}) \\
&= \begin{cases} (-1)^{m-k} (2N - 2m - 1) & \text{for } |k| \leq m \\ (-1)^{m-k+1} (2m + 1) & \text{for } |k| > m \end{cases},
\end{aligned}$$

where we assumed $|k| \leq N$. Thus the characters can be written as

$$\begin{aligned}
\chi_{\mathcal{A}_{k/2N}}(\tau) &= \frac{(-1)^k}{2N} \left[\eta(\tau)^2 + \sum_{l=0}^{|k|-1} (-1)^{l+1} (2l+1) \Lambda_{(2l+1)N, 2N^2}(\tau) \right. \\
&\quad \left. + \sum_{l=|k|}^{N-1} (-1)^l (2N - 2l - 1) \Lambda_{(2l+1)N, 2N^2}(\tau) \right]
\end{aligned}$$

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